QFI QF Model Solutions
Fall 2020

1.

Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

Sources:

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Page 126, 137

Commentary on Question:

Overall, parts (a), (b), and (d) were answered well by most candidates. Part (c) presented some challenge, as did correctly justifying the normal distribution in part (e).

Solution:

(a) Show that X_t satisfies the stochastic differential equation

$$dX_t = \theta f(t) X_t dW_t$$

Commentary on Question:

Most candidates answered this part well. The statement of the question included an extra minus sign in front of the M_t term in the definition of X_t, which was a typo.

From Ito's Formula,

$$dX_t = \frac{\partial X_t}{\partial t}dt + \frac{\partial X_t}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 X_t}{\partial W_t^2} + \cdots = \frac{\partial X_t}{\partial t}dt + \left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)dW_t + \frac{1}{2}\frac{\partial}{\partial W_t}\left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)dt$$

$$= \left[\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial}{\partial W_t}\left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)\right]dt + \left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)dW_t.$$

1. Continued

Now

$$\frac{\partial X_t}{\partial M_t} = \theta \exp\{\theta M_t - \frac{1}{2}\int_0^t f(s)^2 ds\} = \theta X_t$$

and

$$\frac{\partial M_t}{\partial W_t} = f(t)$$

and

$\frac{\partial}{\partial W_t}\left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right) = \theta^2 f(t)^2 X_t.$

Finally,

$$\frac{\partial X_t}{\partial t} = -\frac{1}{2}\theta^2 f(t)^2 X_t$$

resulting in

$$dX_t = \theta f(t) X_t dW_t.$$

(b) Show that $M_t \sim$ Normal(0, $\int_0^t f(s)^2 ds$) for any $t > 0$.

Commentary on Question:
Most candidates attempted the alternative solution but did not receive full credit. Most did not remark that the integrand is deterministic thus implying normality.

Write the preceding in integral form and take expectations:

$$\int_0^t dX_s = \int_0^t \theta f(s) X_s dW_s$$

so

$X_t - X_0 = \int_0^t \theta f(s) X_s dW_s.$

Hence

$$E(X_t) - E(X_0) = E(\int_0^t \theta f(s) X_s dW_s) = 0$$

implying $E(X_t) = E(X_0) = 1$. Thus

$$E\left(e^{\theta M_t}\right) = \exp\left(\frac{1}{2}\theta^2 \int_0^t f(s)^2 ds\right)$$

Hence $M_t \sim Normal(0, \int_0^t f(s)^2 ds)$.

1. Continued

Alternative Solution: As M_t is an Ito integral with deterministic integrand, M_t is normally distributed. Now E[M_t]=0 and Var(M_t)=$\int_0^t f(s)^2 ds$ by Ito isometry.

(c) Show that $Z_t = yt + (1-t)(z + \int_0^t \frac{1}{1-s} dW_s)$ for $0 \le t < 1$.

Commentary on Question:
Candidates struggled on this part of the question. Most used the alternative solution to arrive at the result.

Since $\frac{\partial Y_t}{\partial t} = \frac{y-Z_t}{(1-t)^2}, \frac{\partial Y_t}{\partial Z} = \frac{-1}{1-t}, \frac{\partial^2 Y_t}{\partial Z^2} = 0$, applying Ito's lemma we have

$$dY_t = \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y}{\partial Z} dZ_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial Z^2}(dt)^2$$
$$= \frac{y - Z_t}{(1-t)^2} dt - \frac{dZ_t}{1-t} = -\left(\frac{1}{1-t}\right) dW_t$$

Take integrals to obtain

$$\int_0^t dY_s = -\int_0^t \frac{1}{1-s} dW_s$$

so
$Y_t - Y_0 = -\int_0^t \frac{1}{1-s} dW_s$.

Hence

$$\frac{y - Z_t}{1-t} = y - z - \int_0^t \frac{1}{1-s} dW_s$$

so

$$Z_t = yt + (1-t)\left(z + \int_0^t \frac{1}{1-s} dW_s\right).$$

Alternatively:

$$\frac{dZ_t}{1-t} = \frac{y - Z_t}{(1-t)^2} + \frac{dW_t}{1-t}$$
$$\frac{dZ_t}{1-t} + \frac{Z_t dt}{(1-t)^2} = \frac{y\,dt}{(1-t)^2} + \frac{dW_t}{1-t}$$
$$d\left(\frac{Z_t}{1-t}\right) = \frac{y\,dt}{(1-t)^2} + \frac{dW_t}{1-t}$$
$$\int_0^t d\left(\frac{Z_t}{1-t}\right) = \int_0^t \frac{y\,dt}{(1-t)^2} + \int_0^t \frac{dW_s}{1-s}$$
$$\frac{Z_t}{1-t} - z = \frac{y}{1-t} - y + \int_0^t \frac{dW_s}{1-s}$$

1. Continued

so

$$Z_t = yt + (1-t)\left(z + \int_0^t \frac{1}{1-s} dW_s\right).$$

(d) Find the mean and the variance of Z_t for $0 \le t < 1$.

Commentary on Question
Most candidates were able to derive the mean and variance correctly.

Using the properties of stochastic integrals,

$$E\left(\int_0^t \frac{1}{1-s} dW_s\right) = 0$$

and

$$E\left[\left(\int_0^t \frac{1-t}{1-s} dW_s\right)^2\right] = E\left((1-t)^2 \int_0^t \frac{1}{(1-s)^2} ds\right) = t(1-t)$$

Thus the mean of Z_t is $E(Z_t|Z_0 = z) = yt + z(1-t)$ and the variance is $Var((Z_t|Z_0 = z) = \; t(1-t)$.

(e) Show that Z_t follows a normal distribution for $0 < t < 1$.

Commentary on Question:
Most candidates did not adequately justify normality as it does not follow from being an Ito integral only. Candidates had to comment that the integrand was deterministic.

Now since $\int_0^t \left(\frac{1}{1-s}\right)^2 ds = \frac{t}{1-t} < \infty$ for $0 < t < 1$, $\int_0^t \frac{1}{1-s} dW_s$ is in the form required for part (b), so $\int_0^t \frac{1}{1-s} dW_s$ follows a normal distribution.

Alternatively, noting that $\int_0^t \frac{1}{1-s} dW_s$ is square integrable and the integrand is deterministic allows one to conclude that the Ito integral is normal.

2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1f) Understand and apply Jensen's Inequality.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov's theorem in changing measures.

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

Sources:

Neftci Ch. 12. Pricing Derivative Products: Partial Differential Equations

QFIQ-113-17. Frequently asked questions in quantitative finance 2nd edition

QFIQ-122-20. Equity Indexed Annuities: Downside Protection, But at What Cost?

Commentary on Question:

Most candidates did well on this question. Only a few candidates had difficulties on part (a) and part (c).

Solution:

(a) Describe four critical provisions that are common to equity indexed annuities contracts.

Commentary on Question:

Most candidates were able to list at least four provisions of equity indexed annuities (EIAs). However, a few candidates confused EIAs with variable annuities.

2. Continued

- Minimum guarantee or "floor" value of contract. Each EIA has a minimum guaranteed value to which the investor is entitled.
- Index participation rate. Each EIA declares a rate at which the investor shares in the growth of the reference index (usually the S&P 500).
- The yield-spread cost factor. After calculating the investor's share of index growth, many EIA contracts apply an additional adjustment known as a yield-spread cost factor.
- The growth-rate cap. After applying the participation rate and yield-spread factor adjustments, a typical equity indexed annuity will apply a growth-rate cap.
- Premium bonus credits. Many EIAs offer investors a bonus credit equal to a percentage of the initial premium deposited to the annuity.
- Surrender charges. The existence of surrender charges means that the EIA is not a liquid investment.
- Market-value adjustment. Additionally, many EIAs impose a market-value adjustment if the investor surrenders the contract before the end of its term.

(b) Show that $C \geq f(S_0 - e^{-rT}K)$ using Jensen's inequality.

Commentary on Question:
Almost all candidates did well on this part.

We know that $f(x)$ is a convex function. By Jensen's inequality, we have

$$E^Q[f(S_T - K)] \geq f(E^Q[S_T - K]),$$

which is

$$E^Q[f(S_T - K)] \geq f(E^Q[X_T] + E^Q[Y_T] - K)$$

Since Q is risk-neutral measure, $exp(-rT)\,E^Q[X_T] = X_0$ and $exp(-rT)E^Q[Y_T] = Y_0$.

It follows that

$$\begin{aligned} C = e^{-rT}E^Q[f(S_T - K)] &\geq e^{-rT}f(E^Q[X_T] + E^Q[Y_T] - K) \\ &= f(e^{-rT}E^Q[X_T] + e^{-rT}E^Q[Y_T] - e^{-rT}K) \\ &= f(X_0 + Y_0 - e^{-rT}K) = f(S_0 - e^{-rT}K) \end{aligned}$$

(c) Establish a condition on μ_1, μ_2, σ_1 and σ_2 such that both $X_t e^{-rt}$ and $Y_t e^{-rt}$ are martingales under the risk-neutral measure $\mathbb{Q}$.

Commentary on Question:
This part proved to be the most challenging. Candidates who weren't able to derive the desired condition received partial credit for correct steps.

2. Continued

By product rule, we have

$$d(X_t e^{-rt}) = X_t e^{-rt}((\mu_1 - r)dt + \sigma_1 dW_t) = X_t e^{-rt}\sigma_1\left(\frac{\mu_1 - r}{\sigma_1}dt + dW_t\right)$$

$$d(Y_t e^{-rt}) = Y_t e^{-rt}((\mu_2 - r)dt + \sigma_2 dW_t) = Y_t e^{-rt}\sigma_2\left(\frac{\mu_2 - r}{\sigma_2}dt + dW_t\right)$$

To make both $X_t e^{-rt}$ and $Y_t e^{-rt}$ are martingales under the risk-neutral measure Q, we need to define the following Wiener process under Q

$$dW_t^* = \frac{\mu_1 - r}{\sigma_1}dt + dW_t = \frac{\mu_2 - r}{\sigma_2}dt + dW_t$$

Hence we need

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

(d) Derive the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ by assuming that the condition in part (c) holds.

Commentary on Question:
Candidates did well on this part.

Suppose the condition given in part (d) holds. Let

$$\alpha = \frac{\mu_1 - r}{\sigma_1}$$

and

$$dW_t^* = \alpha dt + dW_t$$

Then by the Girsanov theorem, we have

$$\frac{dQ}{dP} = \xi_T = e^{-\frac{\mu_1 - r}{\sigma_1}W_T - \frac{(\mu_1 - r)^2}{2\sigma_1^2}T}$$

3. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Neftci Chapter 9, 10 (Ito's Lemma) An introduction to the mathematics of Financial Derivatives

Commentary on Question:

Overall, candidates performed below average on this question.

Solution:

(a) Show that for all $i, j = 0, 1, \dots, n-1$

(i) $E\left[\left(\Delta W_{t_i}\right)^4\right] = 3h^2$ using Ito's lemma.

(ii) $E\left[\left(\Delta W_{t_i}\right)^2\left(\Delta W_{t_j}\right)^2\right] = h^2$ if $i < j$.

Commentary on Question:

Candidates performed as expected on part (a). To receive full credit for part (a)(i), Ito's Lemma must be used. To receive full credit for part (a)(ii), independence must be clearly specified or implied.

(i)

From Ito's lemma:

$$\mathrm{d}\left(\left(\mathrm{W_t} - \mathrm{W_{t_i}}\right)^4\right) = 4\left(\mathrm{W_t} - \mathrm{W_{t_i}}\right)^3 \mathrm{dW_t} + 6\left(\mathrm{W_t} - \mathrm{W_{t_i}}\right)^2 \mathrm{dt}.$$

Integrating over $(\mathrm{t_i}, \mathrm{t_{i+1}})$,we have:

$$\left(\mathrm{W_{t_{i+1}}} - \mathrm{W_{t_i}}\right)^4 = 4\int_{\mathrm{t_i}}^{\mathrm{t_{i+1}}} \left(\mathrm{W_t} - \mathrm{W_{t_i}}\right)^3 \mathrm{dW_t} + 6\int_{\mathrm{t_i}}^{\mathrm{t_{i+1}}} \left(\mathrm{W_t} - \mathrm{W_{t_i}}\right)^2 \mathrm{dt}$$

It follows that:

$$\mathrm{E}\left[\left(\Delta \mathrm{W_{t_i}}\right)^4\right] = 0 + 6\int_{\mathrm{t_i}}^{\mathrm{t_{i+1}}} \mathrm{E}\left[\left(\mathrm{W_t} - \mathrm{W_{t_i}}\right)^2\right] \mathrm{dt} = 6\int_{\mathrm{t_i}}^{\mathrm{t_{i+1}}} (\mathrm{t} - \mathrm{t_i})\mathrm{dt}$$
$$= 3(\mathrm{t_{i+1}} - \mathrm{t_i})^2 = 3\mathrm{h}^2$$

(ii)

Since $\Delta \mathrm{W_{t_i}}$ and $\Delta \mathrm{W_{t_j}}$ are independent when $\mathrm{i} < \mathrm{j}$

$$\mathrm{E}\left[\left(\Delta \mathrm{W_{t_i}}\right)^2\left(\Delta \mathrm{W_{t_j}}\right)^2\right] = \mathrm{E}\left[\left(\Delta \mathrm{W_{t_i}}\right)^2\right]\mathrm{E}\left[\left(\Delta \mathrm{W_{t_j}}\right)^2\right] = \mathrm{h}^2$$

3. **Continued**

(b) Show that

$$\frac{W_T^2}{2} - \frac{T}{2} - \sum_{i=0}^{n-1} W_{t_i} \Delta W_{t_i} = \frac{1}{2}\left(-T + \sum_{i=0}^{n-1} \left(\Delta W_{t_i}\right)^2\right)$$

Commentary on Question:
Candidates performed below average on this part.

Using

$$\mathrm{W_{t_i} \Delta W_{t_i}} = \frac{\left(\mathrm{W_{t_i}} + \Delta \mathrm{W_{t_i}}\right)^2}{2} - \frac{\mathrm{W_{t_i}}^2}{2} - \frac{\left(\Delta \mathrm{W_{t_i}}\right)^2}{2} = \frac{\mathrm{W_{t_{i+1}}^2}}{2} - \frac{\mathrm{W_{t_i}}^2}{2} - \frac{\left(\Delta \mathrm{W_{t_i}}\right)^2}{2}$$

It follows that:

$$\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left(\mathrm{W_{t_i} \Delta W_{t_i}}\right) = \frac{\mathrm{W_T}^2}{2} - \frac{\mathrm{W_0^2}}{2} - \sum_{\mathrm{i=0}}^{\mathrm{n-1}} \frac{\left(\Delta \mathrm{W_{t_i}}\right)^2}{2}$$

Hence:

$$\frac{\mathrm{W_T^2}}{2} - \frac{\mathrm{T}}{2} - \sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left(\mathrm{W_{t_i} \Delta W_{t_i}}\right) = \frac{1}{2}\left(-\mathrm{T} + \sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left(\Delta \mathrm{W_{t_i}}\right)^2\right)$$

(c) Show that $\frac{W_T^2}{2} - \frac{T}{2}$ is the mean square limit of the following series

$$V_n = \sum_{i=0}^{n-1} W_{t_i} \Delta W_{t_i}$$

Commentary on Question:
Candidates performed poorly on this part.

From part (b), we have:

$$\mathrm{E}\left[\left(\frac{\mathrm{W_T^2}}{2} - \frac{\mathrm{T}}{2} - \mathrm{V_n}\right)^2\right]$$

$$= \frac{1}{4}\mathrm{E}\left(\left(-\mathrm{T} + \sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left(\Delta \mathrm{W_{t_i}}\right)^2\right)^2\right)$$

$$= \frac{1}{4}\mathrm{E}\left(\left(\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left(\Delta \mathrm{W_{t_i}}\right)^2\right)^2 + \mathrm{T}^2 - 2\mathrm{T}\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left(\Delta \mathrm{W_{t_i}}\right)^2\right)$$

3. **Continued**

$$= \frac{1}{4} E\left(\sum_{i=0}^{n-1} \left(\Delta W_{t_i}\right)^4 + 2 \sum_{0 \le i < j \le n-1} \left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2 + T^2 - 2T \sum_{i=0}^{n-1} \left(\Delta W_{t_i}\right)^2\right)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{n-1} E\left[\left(\Delta W_{t_i}\right)^4\right] + 2 \sum_{0 \le i < j \le n-1} E\left[\left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2\right] + T^2 - 2T \sum_{i=0}^{n-1} h\right)$$

Plug in $E\left[\left(\Delta W_{t_i}\right)^4\right]$ and $E\left[\left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2\right]$ from part (a):

$$\left[\left(\frac{W_T^2}{2} - \frac{T}{2} - V_n\right)^2\right] = \frac{1}{4}(3nh^2 + n(n-1)h^2 + T^2 - 2T^2) = \frac{1}{4}\left(\frac{3T^2}{n} + n(n-1)\frac{T^2}{n^2} - T^2\right)$$

because:

- $\sum_{i=0}^{n-1} E\left[\left(\Delta W_{t_i}\right)^4\right] = n(E\left[\left(\Delta W_{t_i}\right)^4\right] \quad = n3h^2$
- $2\sum_{0 \le i < j \le n-1} E\left[\left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2\right] = n(n-1)E\left[\left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2\right] = n\,(n-1)h^2$
- $2T\sum_{i=0}^{n-1} h = 2Tnh = 2T^2$

Hence:

$$\lim_{n\to\infty} E\left[\left(\frac{W_T^2}{2} - \frac{T}{2} - V_n\right)^2\right] = \frac{1}{4}\lim_{n\to\infty}\left(\frac{3T^2}{n} + n(n-1) \times \frac{T^2}{n^2} - T^2\right) = 0$$

4. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1c) Understand the Ito Integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3e) Demonstrate understanding of option pricing theory and techniques for interest rate derivatives.

(3i) Understand and apply the Heath-Jarrow-Morton approach including the Libor Market Model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch 21

An introduction to the mathematics of t, Neftci Ch. 3, 6, 7, 17

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, Ch 4

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Explain two reasons why a market practitioner would prefer LMM to the Heath-Jarrow-Morton framework for pricing interest rate derivatives.

Commentary on Question:

Candidates had a mixed performance on this question. Few obtained full marks but a majority got partial marks.

4. Continued

Other answers were permitted, but the main 2 come from Veronesi

- BGM/LMM is the specification of the dynamics of LIBOR-based forward rates, which are actually traded instruments in the market vs HJM which is based off of the instantaneous forward rate, so BGM/LMM can be used to price any contract whose cashflows can be decomposed into forward rates.
- BGM/LMM framework is better at handling prices of complex securities under MC simulation.

(b) Critique your colleague's intention to use lognormal distributions for the forward rates and forward swap rates when pricing caps and swaptions.

Commentary on Question:
Candidates did poorly on this question, and those who attempted it mainly listed qualities of the short rate vs. the swap rate. The main point of this question is to note the incompatibility of the 2 assumptions.

Applying the LMM for both swaptions and caps/floors is inappropriate as the assumptions are incompatable and inconsistent with each other.

In particular, for caps/floors, the LMM is used directly on the forward rates and for swaptions on the forward swap rate. However, this is contradictory as the swap rate is a construction of forward rates and the annuity numeraire, but if the forward rate is assumed lognormally distributed (as assumed in pricing caps/floors under LMM) then the forward swap rate cannot be as well.

(c) State the three conditions for a stochastic process to be a martingale

Commentary on Question:
Candidates did extremely well on this part.

S_t is known given I_t that S_t is adapted
E[$|S_t|$] $is\ finite$
E$[S_T|F_t] = S_t\ for\ t < T$

(d) Define $F_{i+1}(t)$ in terms of the zero-coupon bonds $P(t,T_i)$ and $P(t,T_{i+1})$ and explain for which measure/numeraire-pair $F_{i+1}(t)$ is a martingale.

Commentary on Question:
Most candidates were able to obtain partial marks on this question, but very few obtained full marks.

4. Continued

By definition, $F_{i+1}(t) \ = \frac{1}{\tau}\left(\frac{\mathrm{P(t,}T_i)}{\mathrm{P(t,}T_{i+1})} - 1\right) \equiv \frac{\frac{1}{\tau}\left(\mathrm{P(t,}T_i) - \mathrm{P(t,}T_{i+1})\right)}{\mathrm{P(t,}T_{i+1})}$

So, the simply-compounded forward rate is a proportion of longing $\mathrm{P}(\mathrm{t}, T_i)$ and shorting $\mathrm{P}(\mathrm{t}, T_{i+1})$, all normalized by the $\mathrm{P}(\mathrm{t}, T_{i+1})$numeraire. Hence, by construction it is a martingale under the measure $\mathbb{Q}^{T_{i+1}}$.

(e) Show that Z_t satisfies the three conditions in part (c) under $\mathbb{Q}^{T_i}$ using Ito's Lemma.

Commentary on Question:
Candidate performed well on this part. Candidates could answer in a few different ways, including Ito's lemma or directly evaluating the expectation under t, but to obtain full marks, candidates needed to also establishment adaptedness and finiteness of the absolute value of the process.

Define a new stochastic process $H_t = -\int_0^t X_u dW_k(u) - \frac{1}{2}\int_0^t X_u^2 du$

Perform Ito's Lemma on $Z_t \ = e^{H_t}$

$$dZ_t = \frac{\partial Z_t}{\partial t}dt + \frac{\partial Z_t}{\partial H_t}dH_t + \frac{1}{2}\frac{\partial^2 Z_t}{\partial H_t^2}(dH_t)^2$$
$$= Z_t\left(-X_t dW_k(t) - \frac{1}{2}X_t^2 dt\right) + \frac{1}{2}X_t^2 Z_t dt$$
$$= -X_t Z_t dW_k(t)$$

which is a driftless SDE w.r.t. the $\mathbb{Q}^{T_k}$ Brownian motion.

Furthermore, Z_t is clearly $\mathcal{F}_t -$ measurable

$$E_0^{\mathbb{Q}^{T_k}}[|Z_t|] = E_0^{\mathbb{Q}^{T_k}}\left[\left|\exp\left\{-\int_0^t X_u dW_k(u) - \frac{1}{2}\int_0^t X_u^2 du\right\}\right|\right]$$

$$= E_0^{\mathbb{Q}^{T_k}}\left[\exp\left\{-\int_0^t X_u dW_k(u) - \frac{1}{2}\int_0^t X_u^2 du\right\}\right]$$

$$= E_0^{\mathbb{Q}^{T_k}}\left[\exp\left\{-\int_0^t X_u dW_k(u)\right\}\right]\exp\left\{-\frac{1}{2}\int_0^t X_u^2 du\right\}$$

$$= \exp\left\{+\frac{1}{2}\int_0^t X_u^2 du\right\}\exp\left\{-\frac{1}{2}\int_0^t X_u^2 du\right\} = 1 < \infty$$

So all 3 conditions are satisfied

4. Continued

(f) Show that $dG_t = \frac{\tau\sigma_{i+1}F_{i+1}(t)}{1+\tau F_{i+1}(t)} G_t dW_{i+1}(t)$

Commentary on Question:
Very few candidates attempted this question, and only a handful obtained near-full marks.

we have

$$G_t = \frac{P(t,T_i)}{P(0,T_i)}\frac{P(0,T_{i+1})}{P(t,T_{i+1})}$$

$$= \frac{P(0,T_{i+1})}{P(0,T_i)}(1+\tau F_{i+1}(t))$$

Then solving for the SDE we have

$$\begin{aligned}
dG_t &= \tau\frac{P(0,T_{i+1})}{P(0,T_i)} dF_{i+1}(t) \\
&= \tau\frac{P(0,T_{i+1})}{P(0,T_i)}\sigma_{i+1}F_{i+1}(t)dW_{i+1}(t) \\
&= \tau\frac{P(0,T_{i+1})}{P(0,T_i)}\sigma_{i+1}F_{i+1}(t)\frac{1+\tau F_{i+1}(t)}{1+\tau F_{i+1}(t)}dW_{i+1}(t) \\
&= \frac{\tau\sigma_{i+1}F_{i+1}(t)}{1+\tau F_{i+1}(t)}G_t dW_{i+1}(t)
\end{aligned}$$

5. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5d) Demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control.

Sources:

QFIQ-125-20: Guarantees and Target Volatility Funds

On the Importance of Hedging Dynamic Lapses in Variable Annuities, Risk and Rewards, 2015 issue 66 (pp. 12-16)

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Explain the impact on the dynamic rebalancing strategy if the equity volatility estimator overestimates the volatility.

Commentary on Question:

Candidates did well on this part of the question. Most candidates were able to explain the impact of an overestimated equity volatility estimator.

The weight in equity at rebalancing time t is set to:

$w_t^{equity} = \min\left(\frac{\sigma_{target}}{\hat{\sigma}_t^{equity}}, 100\%\right)$ or

$$equity\ ratio = min\left(110\%, \frac{15\%}{realized\ equity\ volatility}\right)$$

where σ_{target} is the target volatility and $\hat{\sigma}_t^{equity}$ is an estimate of the volatility of equity returns.

If the volatility estimator, $\hat{\sigma}_t^{equity}$ is overestimated, we will allocate too little weight in equity, so the fund volatility will fall short of the target.

5. Continued

(b) Under an SVJD model,

(i) Explain the effectiveness of the dynamic rebalancing strategy.

(ii) Explain the impact of volatility jumps on guarantee cost.

Commentary on Question:
Candidates performed well on this question. Most candidates showed an understanding of the implications of the SVJD model.

(i) The existence of jumps in an SVJD model shows that there are sources of equity market volatility which cannot be controlled by the fund's dynamic rebalancing strategy. No matter how we rebalance the fund, the fund volatility is still not perfectly tied to the target.

(ii) Guarantee cost depends on how well target volatility fund can rebalance so that the fund volatility is equal to the target. The presence of jumps significantly complicates the estimate of equity volatility. Using standard dynamic rebalancing strategy, it tends to overweight in equity, at times when equity returns are largest (during a jump). This results in a significant increase in market-consistent guarantee costs relative to those produced by a model without jumps.

(c) Assess how the model results could change if the Heston model were used instead.

Commentary on Question:
Many candidates were able to identify at least two or three changes listed below. However, few candidates received full credit.

Under the Heston model, volatility on average would be lower due to the removal of the impact of the jumps.

With the decreased modeled volatility, the reduction in volatility cost would be expected to be lower for all of the management strategies versus the static 60/40 allocation strategy.

Vega would be expected to be higher for the capped volatility and VIX-indexed fee strategies, as they would not be triggered as much.

Equity allocation would be expected to be higher (in calm markets) for the more active risk-control funds (such as the target volatility and capital preservation strategies) as the reduction mechanism would not be triggered as much due to the lower levels of volatility.

5. Continued

Prospective fees paid in the VIX-indexed strategy would be expected to be lower due to the lower levels of volatility.

(d) Explain how dynamic lapses impact VA fees collected and guarantee cost as the moneyness of the guarantee changes.

Commentary on Question:
Most candidates were able to explain the impact of dynamic lapses on collected fees and guarantee cost for a decreasing moneyness. Few candidates correctly explained the impact of an increasing moneyness on collected fees and guarantee cost.

When equities are rising, the embedded guarantees are less in the money, and policyholders have a stronger incentive to lapse to avoid paying for the guarantees. Lapses reduce the amount of fees collected by the insurance company.

In a down market scenario, the embedded guarantees are more in the money, and policyholders are not as likely to lapse. The increased persistency (more policies) comes at a time when the guarantees are valuable, thus increasing the expected cost of the guarantee to the insurance company.

(e) Assess whether it is advantageous for your company to hedge dynamic lapsation risk if you are unsure about the exact moneyness level at which the policyholder exercises the option to surrender.

Commentary on Question:
Most candidates were able to provide the right assessment and support their answer with sufficient arguments.

Yes, it is advantageous to hedge dynamic lapsation risk, even with the wrong moneyness assumption.

For the Black-Scholes model, when there is no discrepancy between the hedging and market models, we observe that even if the moneyness ratio assumption is set wrong in the hedge, the risk measures are much lower than those obtained in the scenario where dynamic lapsation risk is not hedged at all. In fact, the standard deviation and risk measures in the scenario with a wrong moneyness ratio are approximately twice as large as in a scenario with perfect moneyness assumption, but under a scenario of no dynamic lapsation risk hedge, they are five times larger.

5. Continued

(f) Suggest a management action to lower the lapsation risk.

Commentary on Question:
Most candidates suggested a reasonable action to reduce the lapsation risk.

Add a surrender charge that is intended to discourage lapses, as well as sufficient to cover the loss in income

Reinsure the risk

6. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

(4g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a)

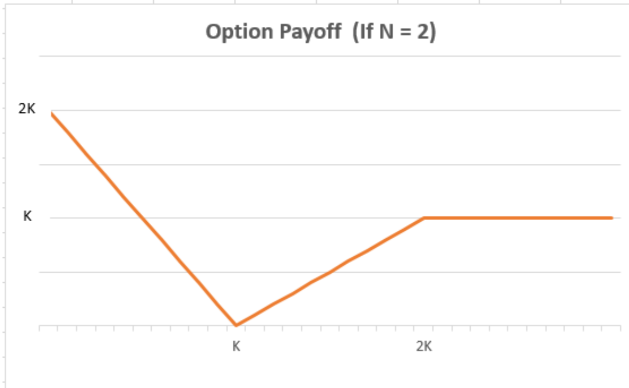
(i) Sketch the payoff graph for the portfolio *B*.

(ii) Construct a static hedging strategy for option *A*, with plain vanilla options and the underlying asset *S*.

Commentary on Question:

Overall, candidates did well on this part of the question. Most candidates were able to graph the option payoffs successfully for portfolio B and construct the hedging strategy for A.

6. Continued



(i)
(ii)

B can be replicated by Long 2 Puts at K, Long 1 Call at K, and Short One Call at 2K. Since B is A – S, we also need to Long 1 Share of S.

(b) Construct a dynamic delta-hedging strategy for this exotic option *A*.

Commentary on Question:
Most candidates knew how to take the derivative of the result in part (a) but were unable to proceed from there to correct the hedging strategy.

From (i), A= C(K) – C(2K) + 2P(K) + S

$$\frac{\partial A}{\partial S} = \frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} + \frac{\partial P(K)}{\partial S} + \frac{\partial S}{\partial S}$$

$$\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} + \frac{2 * \partial[C(K) - S]}{\partial S} + \frac{\partial S}{\partial S}, (Put - Call\ parity)$$

$$(2+1)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - 2 + 1$$

$$(3)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - (1)$$

Delta hedging this option by

- *shorting* $(3)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - (1)$ *unit of underlying asset S*
- *cash balance* $\left[(3)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - (1)\right] S - B$

(c) List pros and cons of static hedging strategies and dynamic hedging strategies.

Commentary on Question:
Most candidates did well on this question. Some candidates did not comment on the availability of assets or list both pros and cons.

6. Continued

Static Hedging Strategy:
Pro:

1. No need to be rebalanced.
2. Do no rely on theoretical models: No assumption for the future behavior of underlying assets, is required.

Con:

1. The options required for hedging strategy might not be available in market.

Dynamic Hedging Strategy:
Pro:

1. More practical in reality, as the strategy can be built with securities available in market.

Con:

1. Require constant rebalancing.
2. Hedging error if the assumptions made for the future behavior of underlying assets deviate from reality.

(d) Show that $rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}$ using the law of one price and Ito's Lemma, where r denotes the constant risk-free rate.

Commentary on Question:
Many candidates did not recognize that the rebalancing factor was also a function of stock price and treated it as a constant in solving this question. Candidates who received full points recognized this fact.

Solve dΣ through Ito lemma

$$d\Sigma = d\left[\alpha\left[V - \frac{\partial V}{\partial s}S\right]\right] = Vd\alpha + \alpha dV - \left[\alpha S d\left(\frac{\partial V}{\partial s}\right) + \frac{\partial V}{\partial s}Sd\alpha + \frac{\partial V}{\partial s}\alpha dS\right]$$

$$\alpha\left[dV - \frac{\partial V}{\partial S}dS\right] + Vd\alpha - \frac{\partial V}{\partial S}Sd\alpha - \alpha S d\left(\frac{\partial V}{\partial S}\right)$$

$$\alpha\left[dV - \frac{\partial V}{\partial s}dS\right]\left[V - \frac{\partial V}{\partial s}S\right]d\alpha = \alpha S d\left[\frac{\partial V}{\partial s}\right])$$

$$\alpha\left[\frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial t}dt - \frac{\partial V}{\partial S}dS\right]$$

$$\alpha\left[\frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial t}dt\right]$$

6. Continued

The portfolio Σ has no risk, as the random facor has been fully hedged. Based on the rule of one price, the return of the portfolio Σ should be equal to risk free rate of r.

$$d\Sigma = \alpha\left[\frac{1}{2}\sigma_S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right]dt = r\Sigma dt = r\alpha\left[V - \frac{\partial V}{\partial S}S\right]dt$$

$$\rightarrow \left[\frac{1}{2}\sigma_S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right] = r\left[V - \frac{\partial V}{\partial S}S\right]$$

$$\rightarrow rV = \frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_s}^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}$$

(e)

(i) Show that the profit and loss function P&L of the hedged portfolio satisfies the following when the hedge is constructed with realized volatility $\sigma_{s,R}$:

$$d(P\,\&\,L) = \frac{1}{2}S^2\Gamma_I\left[{\sigma_{s,R}}^2 - {\sigma_{s,I}}^2\right]dt + (\Delta_I - \Delta_R)\left[(\mu_s - r)Sdt + \sigma_{s,R}SdZ\right]$$

(ii) Determine $d(P\&L)$, if V is hedged with implied volatility $\sigma_{s,I}$ instead.

(iii) Describe key P&L characteristics, when hedging with realized volatility vs. implied volatility, using the results in parts (i) and (ii) to support your answer.

Commentary on Question:
Many candidates were unsure how to complete the first proof or use the result in part (i) to determine the value in part (ii). Most candidates successfully listed several drivers of the P&L using realized and implied volatility.

(i) From (d) $rV = \frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_s}^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}$

$\rightarrow rV = \frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_{s,I}}^2 S^2\Gamma_I + rS\Delta_I$

$\rightarrow \frac{\partial V}{\partial t} = rV - \frac{1}{2}{\sigma_{s,I}}^2 S^2\Gamma_I - rS\Delta_I$ --- *(1)*

Hedged portfolio: V *(long option)* $-\Delta_R S$ *(short* Δ_R *unit of S)* $+ [\Delta_R S - V]$ *(cash balance)*

$dP \wedge L = d[V - \Delta_R S] - [V - \Delta_R S]rdt$ ---- *(2)*

$$dV = \Delta_I\left(dS\middle|\sigma_{s,R}\right) + \frac{\partial V}{\partial t}dt + \frac{1}{2}{\sigma_{s,R}}^2 S^2\Gamma_I dt$$

$\Delta_I\left[\mu_s Sdt + \sigma_{s,R}Sdz\right] + \frac{\partial V}{\partial t}dt + \frac{1}{2}{\sigma_{s,R}}^2 S^2\Gamma_I dt$ ---- *(3)*

$d\Delta_R S = \Delta_R\left(dS\middle|\sigma_{s,R}\right) = \Delta_R\left[\left[\mu_s Sdt + \sigma_{s,R}Sdz\right]\right]$ ---- *(4)*

6. Continued

(1), (3), (4) → (2)

$$dP \wedge L = \Delta_I\left[\mu_s S dt + \sigma_{s,R} S dz\right] + \left[rV - \frac{1}{2}\sigma_{s,I}{}^2 S^2 \Gamma_I - rS\Delta_I\right]dt$$
$$\frac{+1}{2}\sigma_{s,R}{}^2 S^2 \Gamma_I dt - \Delta_R\left[\left[\mu_s S dt + \sigma_{s,R} S dz\right]\right]$$
$$-[V - \Delta_R S]rdt$$

This can be simplified and rearranged to

$$d(P \& L) = \frac{1}{2}S^2\Gamma_I\left[\sigma_{s,R}{}^2 - \sigma_{s,I}{}^2\right]dt + (\Delta_I - \Delta_R)\left[(\mu_s - r)Sdt + \sigma_{s,R} SdZ\right]$$

(ii) From i), if hedged with implied volatility, $\Delta_I = \Delta_R$ *so the last term cancels leaving*

$$\frac{1}{2}\left(\sigma_{s,R}{}^2 - \sigma_{s,I}{}^2\right)S^2\Gamma_I dt$$

(iii)

Realized volatility:

- *Know exactly what profit to get at expiration.*

Hedged portfolio

=V (long option) $-\Delta_R S$ (short Δ_R unit of S) + $[\Delta_R S - V]$ (cash balance)

=V_I (priced with $\sigma_{s,I}$) - V_R (replicated with $\sigma_{s,R}$) + Cash Balance

- *The P&L could fluctuate during the life of option.*

dP&L contains a stochastic term $(\Delta_I - \Delta_R)\ \sigma_{s,R} S dz$

Implied volatility:

- *No fluctuation in P&L during the life of option.*

  dP&L $= \frac{1}{2}\left(\sigma_{s,R}{}^2 - \sigma_{s,I}{}^2\right)S^2\Gamma_I dt$, which has no stochastic term.

- *Make profit as long as on the right side of trade. (i.e. long option if $\sigma_{s,R}{}^2 > \sigma_{s,I}{}^2$)*

dP&L $= \frac{1}{2}\left(\sigma_{s,R}{}^2 - \sigma_{s,I}{}^2\right)S^2\Gamma_I dt > 0$, given the gamma of option ($\Gamma_I > 0$)

- *Unable to predict how much money you will make.*

7. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016, Ch. 3,6,7,8,14,18

Commentary on Question:

This question was intended to measure candidates' understanding of equity derivative and volatility smile. Most candidates well understood the volatility smile but showed a lack of knowledge of the practical equity derivative application.

Solution:

(a)

(i) Explain volatility smiles.

(ii) Compare the common two approaches that describes the volatility smiles.

(iii) Explain which approach was provided by the junior analyst.

Commentary on Question:

Most candidates performed well in part (a). The purpose of part (a) was to measure the understanding of volatility smile.

(i) When plot the market implied volatility vs the strike price or in-the-moneyness of the options, we often observe implied volatility is non-constant as function of strikes, with lower implied volatility near the at-the-money (ATM) strike, and higher implied volatility for both lower and higher strikes.

7. Continued

(ii) Volatility smiles are generally described in two ways:

1) sticky strike, where an option with a fixed strike will always have the same implied volatility, $\Sigma(S,K) = \Sigma_0 - \beta(K\text{-}S0)$
2) sticky delta, where an option's volatility depends only on its in-the-moneyness K/S. $\Sigma(S,K) = \Sigma_0 - \beta(K\text{-}S)$

(iii) The junior assistant most likely described sticky strike, as volatility is only related to the strike level.

(b) Describe the most salient characteristics of the equity volatility smile.

Commentary on Question:
Candidates demonstrated modest understanding in part (b).

- Its most notable character is the negative slope as a function of the strike.
- The negative slope is generally steeper for short expiration.
- Implied volatility and index returns are negatively correlated.
- Equity smile is often a smirk than a smile – increase and decrease in implied volatility are often asymmetric. skew is partially due to an asymmetry in the way equity index movement: large negative returns are much more frequent than large positive returns.
- There is also a demand component that contributes to smile, people are willing to pay additional premium for hedge of large movement.

(c) Identify the trades of the replicating portfolio.

Commentary on Question:
Most candidates performed poorly in part (c). Some candidates showed a lack of understanding of replicating portfolio construction.

By offering return of premium and a cap on T&T growth, the index annuity longs an at-the-money call and shorts an out-of-the-money call at 5 delta.

Assume no lapse or redemption before renewal, to hedge this liability, company should buy an at-the-money call and sell an out-the-money call at 105%, at inception.

(d) Calculate the price for the replicating portfolio and determine whether the budget is sufficient for the hedging, using the fitted implied volatility function IV(K) provided.

7. Continued

Commentary on Question:
The candidates who had a right approach in part (c) also performed well in part (d). However, many candidates made mistakes in calculation.

$$C(S,K,t,\sigma,r) = SN(d_1) - Ke^{-rt}N(d_2)$$
$$d_{1,2} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{\tau}}$$
$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,dy$$

For at-the-money call: S=100, K=100, r =3%, d=0%, t=1, σ=15% + (100-100) *1.4%=15%
Plug into formula above, C(100) = 7.49

For Out-of-money call: S=100, K=105, r =3%, d=0%, t=1; σ=15% + (105-100) *1.4%=22%
Plug into formula above, C(105) = 7.93

The hedge portfolio price = 7.49 – 7.93 = -0.44 < 0.5
Yes, the budget is sufficient

(e) Explain the reasonableness of the implied volatility function IV(K) in the context of smile arbitrage.

Commentary on Question:
Most candidates performed poorly in part (e). Most candidates didn't approach the question from the arbitrage-free perspective.

When a portfolio of options with non-negative (non-positive) payoff actually has a negative (positive) market price while using the volatility smile, the volatility smile is considered not arbitrage-free.

Since we are long an at-the-money call (paying \$5.80) and short an out-the-money call (receiving \$6.50), we are receiving \$0.7 by constructing this portfolio, while the portfolio will have a non-negative payoff (call spread).

Therefore, arbitrage exists, due to unreasonable volatility smile.

(f)

(i) Identify types of market conditions that would negatively affect the ability to manage the product with the added guarantee.

(ii) Suggest a modeling approach to better measure the risk.

7. Continued

(i) The embedded option in this index annuity is basically a call spread (long ATM call + short OTM call), with the new product design feature, the price of the portfolio is 0.5 = ATM call – OTM call, where we could back out the OTM call strike, however which is floored at 3%.

The possible challenge of such product design is budget is not enough to offer a cap of at least 3%, then the product has to be offered at a loss or below the expected profit level.

(ii) It is important for the pricing actuaries to understand the market condition where such risk exists.

This could be achieved by model the market condition: interest rate, equity level, equity volatility stochastically, especially equity volatility smile, as it drives the difference between ATM call and OTM call.

With, stochastically volatility models, volatility can change through time, are a function of time, index level, strike level.

Pros: automatically create a volatility smile – is appropriate for pricing exotic option, it could also match the term structure of the volatility.

Cons: cannot replicate European options, only can approximate. The calibration can be unstable, resulting in jumps in mark-to-market Profit/Loss; can be calibrated using vanilla option or exotic option, but not both at the same time.

8. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(3b) Understand and apply various one-factor interest rate models.

Sources:

1) Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 15, 19)

2) An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 9, 10)

3) Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (page 112,116,136, 159)

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Explain why interest rates are always positive in this model.

Commentary on Question:
Candidates generally answered this question okay. The most common error or omitted portion were not commenting on the technical condition was true to ensure the drift term cannot force it negative.

When the interest rate r_t is moving toward zero, the diffusion part $\sqrt{\alpha r_t}$ declines, and it becomes in fact zero when r_t hits zero.

When r_t = 0, the only term left is $dr_t = \gamma(\bar{r})$>0. Thus, the next step will be for sure that r_t increases (because the change $dr_t > 0$).

8. Continued

One important caveat is that to ensure the interest rate process is always positive (and well defined), we must have the following technical condition satisfied: $\gamma\bar{r} > \frac{1}{2}\alpha$

That is, the term that "pulls up" the interest rate when r_t hits zero, " $\gamma\bar{r}$," must be large enough.

(b) Show that $r_t = e^{-\gamma t}r_0 + \bar{r}(1-e^{-\gamma t}) + \sqrt{a}e^{-\gamma t}\int_0^t e^{\gamma s}\sqrt{r_s}\,dX_s$.

Commentary on Question:
Candidates performed well on this question. Clearer answers were those that stated they were using Ito's lemma and mentioned integrating both sides.

Using the Ito`s lemma on $Z_t = e^{\gamma t}r$, we have

$$d(e^{\gamma t}r_t) = \frac{\partial}{\partial t}(e^{\gamma t}r_t)dt + \frac{\partial}{\partial r}(e^{\gamma t}r)|_{r=r_t}dr_t + \frac{1}{2}\frac{\partial^2}{\partial r^2}(e^{\gamma t}r)|_{r=r_t}\ (dr_t)^2$$

$$= \gamma e^{\gamma t}r_t dt + e^{\gamma t}[(\gamma(\bar{r}-r_t)dt + \sqrt{\alpha r_t}dX_t]$$

$$=\gamma(\bar{r})e^{\gamma t}\ dt + e^{\gamma t}\sqrt{\alpha r_t}dX_t$$

Integrating the above expressions on both sides from 0 to t

$$\int_0^t d(e^{\gamma s}r_s) = \int_0^t \gamma(\bar{r})e^{\gamma s}\ ds + \int_0^t e^{\gamma s}\sqrt{\alpha r_s}dX_s$$

$$r_t = e^{-\gamma t}r_0 + \bar{r}(1-e^{-\gamma t}) + \sqrt{\alpha}\ e^{-\gamma t}\int_0^t e^{\gamma s}\sqrt{r_s}\ dX_s$$

(c) Determine $E[r_t]$ and $Var[r_t]$.

Commentary on Question:
Candidates performed okay on this part of the question. Most had no issue with the Expectation and the beginning of the Variance including the relation with Ito's Isometry. Most struggled with substituting back in the E[t] to push to the final equation.

$$E[r_t] = e^{-\gamma t}r_0 + \bar{r}(1-e^{-\gamma t}) + E[\sqrt{\alpha}\ e^{-\gamma t}\int_0^t e^{\gamma s}\sqrt{r_s}\ dX_s]$$

$$E[r_t] = e^{-\gamma t}r_0 + \bar{r}(1-e^{-\gamma t})\ as\ E[\sqrt{\alpha}\ e^{-\gamma t}\textstyle\int_0^t e^{\gamma s}\sqrt{r_s}\ dX_s] = 0$$

8. Continued

$$Var[r_t] = E[r_t - E(r_t)]^2 = \alpha\, e^{-2\gamma t} E[\left(\int_0^t e^{\gamma s}\sqrt{r_s}\, dX_s\right)^2]$$

using Ito's Isometry

$$= \alpha\, e^{-2\gamma t}\int_0^t e^{2\gamma s}\, E[r_s]ds$$

Substituting $E[r_t]$ *gives:*

$$\alpha\, e^{-2\gamma t}\int_0^t e^{2\gamma s}\,[e^{-\gamma s}r_0 + \bar{r}(1-e^{-\gamma s})]\, ds$$
$$= (\alpha\,\bar{r})\frac{1}{2\gamma}(1 - e^{-2\gamma t} - 2e^{-\gamma t} + 2e^{-2\gamma t}) + \frac{\alpha}{\gamma}r_0 e^{-2\gamma t}(e^{\gamma t} - 1)$$

or

$$\frac{\alpha\bar{r}}{2\gamma}(1 - e^{-\gamma t})^2 + \frac{\alpha}{\gamma}\, r_0 e^{-\gamma t}(1 - e^{-\gamma t})$$

(d) Explain whether the CIR model belongs to the class of generalized affine models.

Commentary on Question:
Candidates faired okay on this part of the question. Simply stating the model was affine without explanation received no credit. Good solutions explicitly laid out how the conditions of the generalized model were satisfied. Credit was also given for the specific form of zero-coupon bonds under generalized affine models.

Generalized affine model has the general form

$$dr_t = (\theta_t - \gamma_t\, r_t)dt + \sqrt{\sigma_t^2 + \alpha_t r_t dX_t}$$

When $\sigma_t = 0$ *and* $\alpha_t = \alpha$ *is a constant,* $\gamma_t = \gamma, \theta_t = \gamma\,\bar{r} > \frac{1}{2}\alpha,\ \alpha > 0$ *we obtained the generalized Cox, Ross Ingersoll model.*

As such the Cox, Ingersoll and Ross model belongs to the class of generalized affine models.

(e) Express $\frac{\partial Z}{\partial t}$, $\frac{\partial Z}{\partial r}$ and $\frac{\partial^2 Z}{\partial r^2}$ in terms of $Z(r_t, t, T)$, $A(t,T)$ and $B(t,T)$.

Commentary on Question:
Candidates performed well on this section. Most common mistake was with the $\frac{\partial Z}{\partial t}$ *term either including an incorrect extra term or omitting the r term on* $\frac{\partial B}{\partial t}$ *component.*

8. Continued

$Z(r,t,T) = e^{A(t,T)-B(t,T)r}$

$$\frac{\partial Z}{\partial t} = \left[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r\right] Z(r,t,T)$$

$$\frac{\partial Z}{\partial r} = -B(t,T)\, Z(r,t,T)$$

$$\frac{\partial^2 Z}{\partial r^2} = B^2(t,T)\, Z(r,t,T)$$

(f) Show that

(i) $$\frac{\partial A}{\partial t} = \gamma \bar{r} B(t,T)$$

(ii) $$\frac{\partial B}{\partial t} = \gamma B(t,T) + \frac{1}{2} a B(t,T)^2 - 1$$

Commentary on Question:
Candidates performed poorly on this question. A good portion of the candidate did not attempt this part of the question. An alternate solution was also accepted using given formulas and is present below.

Primary Solution:

Plugging the results of part (e) to the fundamental pricing equation

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r}\gamma(\bar{r} - r) + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2} r\alpha = rZ$$

we have

$$\left[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r\right] Z - BZ\,\gamma(\bar{r} - r) + \frac{1}{2}B^2 Z\, r\alpha = rZ$$

It follows that

$$\left[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r\right] - B(t,T)\,\gamma(\bar{r} - r) + \frac{1}{2}B^2(t,T) r\alpha = r$$

Rearranging the terms

$$\left[\frac{\partial A}{\partial t} - B(t,T)\,\gamma(\bar{r})\right] - \left(\frac{\partial B}{\partial t} - B(t,T)\,\gamma - \frac{1}{2}B^2(t,T)\alpha + 1\right) r = 0$$

In order to have the above expression =0 for all t and r

8. Continued

$\frac{\partial A}{\partial t} - B(t,T)\,\gamma(\bar{r}) = 0$ *This implies* $\frac{\partial A}{\partial t} = B(t,T)\,\gamma(\bar{r})$

$$\left(\frac{\partial B}{\partial t} - B(t,T)\,\gamma - \frac{1}{2}B^2(t,T)\alpha + 1\right) = 0$$

This implies

$$\frac{\partial B}{\partial t} = \gamma B(t,T) + \frac{1}{2}\alpha B^2(t,T) - 1$$

Alternative Solution:

Realizing that this is a generalized affine model and the formula sheet gives the equations for the following:

(19.59) $Z(r,t,T) = e^{A(t,T)-B(t,T)r}$

(19.60) $\frac{\partial B}{\partial t} = B(t,T)\gamma_t + \frac{1}{2}B(t,T)^2\alpha_t - 1$

(19.61) $\frac{\partial A}{\partial t} = B(t,T)\theta_t - \frac{1}{2}B(t,T)^2\sigma_t^2$

Using the solution to part (d)

When $\sigma_t = 0$ *and* $\alpha_t = \alpha$ *is a constant,* $\gamma_t = \gamma, \theta_t = \gamma\,\bar{r} > \frac{1}{2}\alpha,\ \alpha > 0$ *we obtained the generalized Cox, Ross Ingersoll model.*

Since $\theta_t = \gamma\,\bar{r}$ *and* $\sigma_t = 0$*, substituting we get*

$$\frac{\partial A}{\partial t} = \gamma\bar{r}B(t,T)$$

And similarly since $\gamma_t = \gamma$ and $\alpha_t = \alpha$, we get

$$\frac{\partial B}{\partial t} = \gamma B(t,T) + \frac{1}{2}aB(t,T)^2 - 1$$

9. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.

(2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Veronesi, Fixed Income Securities (Chapters 3, 4)

Commentary on Question:

The description of the immunization and the cash flow strategies to meet the annuity commitment, and the calculation of the cash flows after the first 6-month period were generally well done, to the exception of some technical aspects that will be described in the appropriate part below.
An aspect that could be enhanced and that usually leads to obtaining additional credits, is to give more details when asked to describe or explain a concept. Those details are contained in the reading material and they should not be ignored when asked about an important to know actuarial issue. Examples of this will also be provided below.

Solution:

(a) Describe an immunization strategy that you can implement to meet the annuity commitment.

Commentary on Question:
Some candidates forgot to include a cash component of the portfolio, and they relied on the fact that a 100% investment in the 30-year bond, with its $20,000 semi-annual coupon, would meet the annuity payment. They were so not able to calculate the percentages to be invested in the bond, and in cash, based on the relation between the duration of the annuity and the duration of the portfolio assets. Also, an important step of the immunization strategy is the necessity of rebalancing the portfolio at each future date. Only a few candidates made that reference.

An immunization strategy can be implemented as follows:

1. **Split the $1 million deposit received at initiation into to two investments**: one is invested in cash receiving overnight deposit rate, and the other is invested in the 30-year bond receiving the fixed 4% coupon paid semiannually.

9. Continued

2. **The percentage of investment in long-term bonds is determined by the ratio between the duration** of the annuity (13.91 initially) **and the duration** of the 30-year bond (19.93 initially). The allocation corresponds to the percentage to be invested in the 30-year bond. The immunization strategy equates the duration of the bond with the duration of the annuity, thus immunizing interest rate risk.

$$\text{Percentage investment in long-term bond} = x_t = \frac{\text{Duration of annuity}}{\text{Duration of long-term bond.}}$$

3. Every six months upon receiving the bond coupon payment, **rebalance** the total portfolio (including the floating payments from the cash investment) between cash and the 30-year bonds and the split ratio is adjusted to reflect the change in duration of the 30-year bond and the annuity.

(b) Determine the cash flows after six months, assuming that the overnight deposit rate stays the same during the period. Explain why the immunization strategy would work.

Commentary on Question:
The calculation of the cash flows was generally done successfully except for those who ignored the cash portion of the investment. This induced some errors in either the calculation of the interest cumulated over 6 months, or the calculation of the coupon payment on the 30-year bond

Percentage Investment in long-term bond:

$$x_0 = \frac{Duration\ of\ annuity}{Duration\ of\ 30\text{−}year\ bond} = \frac{13.91}{19.93} = 70\%$$

1. Collect the interest cumulated over the six month on the cash investment;

$$= W_t \times (1 - x_t) \times r_t/2$$

2. Collect the coupon payment on the 30-year bond (being valued at par);

$$= \frac{W_t \times x_t}{\text{Price T-bond in Column (6)}} \times 4\%/2$$

= $1,000,000 * 70% * 4% / 2

= $13,961

Pay out the annuity cash flow of $20,000

9. **Continued**

(c) Explain why the immunization strategy would work.

Commentary on Question:
Here is an example where a more detailed description of the concept of immunization would have been appropriate. As per the study material, describing why having the portfolio invested 100% in cash, or alternatively 100% in the 30-year bond, would not be the solution based on the impact or rising/decreasing interest rates.

Also, very few candidates referred to the positive convexity of a 30-year bond that would lead to a positive expected return for small interest rate changes.

Why the immunization strategy works:

1. Allocating 100% to the 30-year bond loses money when interest rates go up, because the bond price decreases when the interest rate rises.

2. Allocating 100% to cash loses money when the interest rate declines. If the interest rate goes to zero, then there is not enough capital to make up the annuity coupon.

3. The immunization strategy is in the middle, which effectively ensures that the losses on the cash investment due to declining interest rates are compensated by the capital gain from the 30-year bond and the losses on the 30-year bond due to rising interest rates are compensated by the higher floating payments tied to the higher overnight deposit rates.

4. The duration matching ensures the interest rate sensitivity is under control.

5. In addition, it works because the 30-year coupon bond has a positive convexity, and a delta hedged positive convexity portfolio (assets minus liability) should have a positive expected return for small interest rate changes.

(d) Explain whether Cash Flow Matching is a viable strategy. Justify your conclusion by identifying its drawbacks or benefits in this context.

Commentary on Question:
Here again, in describing the exact matching of the future cash flows it was expected that the candidates point to the use of zero coupons bonds for the 60 payment dates of the annuity as being as a good theorical and direct way to do it, subject for sure to the appropriate constraints.

9. **Continued**

As a cash-flow matching approach, many suggested to use the fixed $20,000 coupon from the 30-year bond as exactly matching the cash flows. But they then ignored the maturity value of the bond for which one can borrow/enter a derivative that cancels out the principal payment.

Cash Flow Matching is not a viable solution in this context.

- To exactly match the future cash flows, one way that the financial institution has to do is to purchase 60 zero coupon bonds, each with $20,000 face value, and with maturities of 6 months, 1 year, 1.5 year, and so on, up to 30 years. There are other matching solutions: For example, buying a 30-year coupon bond that matches all the payments and borrowing/entering a derivative that cancels out the principal payment is another way to achieve that.
- Economically speaking (from the pricing perspective), this should be theoretically doable.
- Such a sequence of zero-coupon bonds might not be fully available in the market.
- Even if they are available, they cannot be implemented due to transaction costs and liquidity issues.

10. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4f) Appreciate how hedge strategies may go awry.

(4l) Explain various issues and approaches for fitting a volatility surface.

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
(i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
(ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

(5e) Demonstrate an understanding of how differences between modeled and actual outcomes for guarantees affect financial results over time.

Sources:
QFIQ-128-20: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:
This question is to test candidates' understanding on the features of guaranteed riders of a VA contract and hedging strategies against equity, interest rate and volatility risk.

Solution:
(a) Calculate the position in each of the three assets at time 0.

Commentary on Question:
A few candidates answered this question and very few of them successfully identify the position. Most candidates left this part as blank.

10. Continued

With $L_0 = 0$, $\prod_0$ (for a perfect hedge) should be 0:

$0 = \Delta_t * 200 + n_t * 100 - 500M$ (1)

The self-financing hedge portfolio should satisfy the following:

$$\Pi_{t+h} = \left(\Pi_t - \Delta_t S_t - n_t P_{t,t+T^B}\right) B_{t+h}/B_t + \Delta_t S_{t+h} + n_t P_{t+h,t+T^B}.$$

$4.5M = (0 - \Delta_t * 200 - n_t * 100) * 1.005 + \Delta_t * 203 + n_t * 101$
And
$507M = \Delta_t * 203 + n_t * 101$ (2)

By solving the two equations, we can get Δ_t = 2M and n_t= 1M

So,

- A position of 2M in stock
- A position in 1M in zero-coupon bond
- A position in bank account with total borrowing of $500M such that the hedging portfolio is self-financing

(b) Define the objective of the hedging strategy in terms of the insurer's hedged loss at maturity.

Commentary on Question:
More than half of candidates performed well in this question. The remaining candidates failed to correctly describe that the objective of a hedging strategy is to offset the insurer's unhedged loss at maturity with the terminal value of the hedging portfolio.

The insurer's hedged loss at maturity is: $HL_t = L_t - \Pi_t$
The objective of the hedging portfolio is to offset L_t, and therefore to result in a hedged loss of approximately zero at maturity

(c) State one problem with using the forward-looking approach to calibrate the stock volatility.

Commentary on Question:
Only a few candidates successfully identified relevant problems that describe the difficulties of using the forward-looking approach to calibrate the stock volatility. Candidates should compare the differences in features between VA contracts and traded derivatives in market (outlined in the solution below).

10. Continued

The problems of the approach include:

- VAs have long-term maturities, while forward-looking measures are extracted from shorter-term traded options (which may involve unsound extrapolation)
- Two models that are well calibrated to the implied volatility vanilla option surface may lead to very different prices and hedge ratios for exotic option
- Therefore, there is no guarantee that implied volatilities from traded vanilla options will consist in appropriate volatility inputs when hedging VAs with non-vanilla features, such as GMWBs

(d)

(i) Identify the sources of model risk in your hedging strategy under each of Models A, B, and C.

(ii) Identify the corresponding market model by matching Model X, Y, and Z to Model A, B, or C. Justify your answer.

Commentary on Question:
For candidates who answered this question, candidates performed well in part (i). They successfully identified the features of those three models. They also performed fairly in part (ii) but some of them didn't provide any justification on the matches, and some of them didn't correctly understand the relationship between the impact of model risk on the effectiveness of hedging strategies and the level of resulting CTE.

(i)
Model A: difference in interest rate model (CIR vs Vasicek)
Model B: changes in the slope and curvature of term structure are not accounted for in the hedging strategy, but are reflected in the model
Model C: stochastic volatility and change in the slope and curvature of term structure are not accounted for in the hedging strategy, but are reflected in the model

(ii)
Model X: CTE 95% of 1.8 is for Model B.
Model Y: CTE 95% of 0.5 is for Model A.
Model Z: CTE 95% of 4 is for Model C.

Justification:
Since the insurer always uses the BSV model to establish its hedging, the three data-generating models give rise to varying degrees of model risk. Market model with higher level of deviation from the BSV model will get the less effective hedging results.

10. Continued

(e) Explain whether you agree with the student's result.

Commentary on Question:
Only a few candidates successfully identified the student is wrong. For some of those who disagreed with the students, they failed to provide appropriate justifications. Successful candidates noticed the hedge strategy didn't hedge market volatility and considered its impact on hedged loss.

Do not agree with the result.

Because this hedge does not protect against Vega risk, the hedged loss should not be mostly centered around zero for varying degrees of stock market volatility. The hedged loss should be centered at zero around the unconditional volatility with a trend line for varying degrees of stock market volatility.

(f)

(i) Explain how a delta-only hedging strategy would affect the insurer's hedged loss if your expectation becomes a reality.

(ii) Explain how a wrong expectation would affect the insurer's hedged loss after modifying the hedge strategy.

Commentary on Question:
About a half of candidates performed well in this question. The remaining candidates failed to understand the impact of interest rate on the VA guaranteed riders

(i) If the interest rates rise steadily throughout the term of the VA contracts, the value of the guarantees offered by the insurer will decrease, resulting in a net gain for the insurer if rho risk is not hedged (i.e., delta-only strategy).

(ii) If interest rates turn out to be low and stable, a delta-rho hedge strategy can reduce the insurer's exposure to large hedging losses, as compared to delta-only hedge.

11. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(4b) Identify limitations of the Black-Scholes-Merton pricing formula

(4f) Appreciate how hedge strategies may go awry.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

Sources:

Volatility Smile Chapters 4, 19

The Impact of Stochastic Volatility on Pricing, Hedging and Hedge Efficiency of Withdrawal Benefit Guarantees in Variable Annuities Nefci Ch. 6

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Explain why you would expect ρ to be positive or negative.

Commentary on Question:

Candidate did well on this part by explaining the reverse correlation between the stock market and volatility.

11. Continued

Under the BSM model with constant volatility σ,

$$\frac{\partial C_{BSM}}{\partial S_t} = N(d_1)$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + {}^1\!/_2\,\sigma^2 T}{\sigma\sqrt{T}}$$

Thus,

$$\frac{\partial^2 C_{BSM}}{\partial S_t \partial \sigma} = \frac{\partial N(d_1)}{\partial \sigma} = \phi(d_1)\left(\frac{1}{2}\sqrt{T} - \frac{\ln\left(\frac{S_t}{K}\right)}{\sigma^2\sqrt{T}}\right)$$

So vanna tends to be positive when the call is out of money ($K > S_t$), and negative when the call is in the money.
Given

$$dC = \frac{1}{2}\frac{\partial^2 C_{BSM}}{\partial \sigma^2} E[d\sigma_t^2] + \frac{\partial^2 C_{BSM}}{\partial S_t \partial \sigma} E[dS_t d\sigma_t]$$

If $E[dS_t d\sigma_t] > 0$ (i.e. $\rho > 0$), then dC tends to be driven up (actual call price is above the BSM value) by the second term when the call is out of money and driven down (actual call price below the BSM value) when the call is in the money. E.g. implied volatility is high at high strikes and low at low strikes. However, the volatility smile of an equity index usually has negative skew. Thus we would expect ρ to be negative.

Alternatively: Points are awarded for the explanation using down markets (decrease of X) and higher vols (increase of sigma^2) based on negative skew.

(b)

(i) Show that $d\left(E_0^{\mathbb{Q}}\left[\sigma_t^2\right]\right) = k\left(\theta^2 - E_0^{\mathbb{Q}}\left[\sigma_t^2\right]\right)dt$.

(ii) Show that $E_0^{\mathbb{Q}}\left[\sigma_t^2\right] = e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2$.

(iii) Calculate σ_K^2 in terms of k, θ, σ_0, and T.

Commentary on Question:
Most candidates did well on (i) and (iii), but struggled on (ii),

(i)
Since $d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dY_t$, thus

$$E(d\sigma_t^2) = E[k(\theta^2 - \sigma_t^2)dt] + E(\gamma\sigma_t dY_t)$$

11. Continued

Since W_t^2 is a standard Wiener process, $E(\gamma\sigma_t dY_t) = 0$, thus

$$E(d\sigma_t^2) = E[k(\theta^2 - \sigma_t^2)dt] = k\theta^2 dt - kE[\sigma_t^2 dt]$$
$$dE(\sigma_t^2) = k\theta^2 dt - kE[\sigma_t^2]dt$$

(ii) – solution of ODE in part (i)

$$d[e^{kt}E(\sigma_t^2)] = e^{kt}dE(\sigma_t^2) + ke^{kt}E(\sigma_t^2)dt = ke^{kt}\theta^2 dt - ke^{kt}E[\sigma_t^2]dt + ke^{kt}E(\sigma_t^2)dt$$
$$= ke^{kt}\theta^2 dt$$

Integrating both sides

$$e^{kt}E(\sigma_t^2) - \sigma_0^2 = \int_0^t ke^{ks}\theta^2 ds = \theta^2 e^{kt} - \theta^2$$
$$E(\sigma_t^2) = \theta^2 - \theta^2 e^{-kt} + \sigma_0^2 e^{-kt} = e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2$$

(iii)

$$\sigma_K^2 = \frac{1}{T}\int_{t=0}^{T}[e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2]dt = \frac{1}{T}\left[\frac{(1 - e^{-kT})(\sigma_0^2 - \theta^2)}{k} + \theta^2 T\right]$$
$$= \frac{(1 - e^{-kT})(\sigma_0^2 - \theta^2)}{kT} + \theta^2$$

(c)

(i) Show that $\sigma_R^2 = \frac{2}{T}\left[\int_0^T \frac{1}{S_t} dS_t - ln\frac{S_T}{S_0}\right]$.

(ii) Describe a continuous replication strategy using S_t and a portfolio $\pi\left(S, S^*\right)$ with the appropriate choice of S^* to capture the realized variance σ_R^2.

Commentary on Question:
This question tests the understanding and application of Ito's Lemma. Most candidates successfully derived the formula for realized variance from time 0 to T, but some could not explain very well the replicating portfolio. Very few candidates attempted part (c)(ii).

(i) Since

$$\frac{dS_t}{S_t} = rdt + \sigma_t dX_t$$

Thus by Ito's lemma,

$$dlnS_t = (r - \frac{\sigma_t^2}{2})dt + \sigma_t dX_t$$

11. Continued

Subtracting the two equations to eliminate the rdt and dX_t term,

$$\frac{dS_t}{S_t} - dlnS_t = \frac{\sigma_t^2}{2}dt$$

Integrating both sides, the annualized realized variance from time 0 to T can be expressed as:

$$\frac{1}{T}\int_{t=0}^{T}\sigma_t^2 dt = \sigma_R^2 = \frac{2}{T}\left[\int_{t=0}^{T}\frac{1}{S_t}dS_t - ln\frac{S_T}{S_0}\right]$$

(ii) This equation represents a replicating portfolio that captures the realized variance from time 0 to time T:

1. Continuously rebalance to long $\frac{1}{S_t}$ share of the stock S_t.
2. Static short position in a contract that pays log return of the stock at time T.

Now need to replicated the 2nd position above with the portfolio of calls and puts.
The given portfolio $\int_0^{S^*}\frac{1}{K^2}P(K,T)dK + \int_{S^*}^{\infty}\frac{1}{K^2}C(K,T)dK$ has the following pay off at maturity time T:

- If $S_T > S^*$, then all of the puts will expire worthless, and calls with strike price $K < S_T$ will have a total payoff:

$$\int_{S^*}^{S_T}\frac{S_T - K}{K^2}dK = \frac{S_T - S^*}{S^*} + ln\frac{S^*}{S_T}$$

- If $S_T < S^*$, then all of the calls will expire worthless, and puts with strike price $K > S_T$ will have a total payoff:

$$\int_{S_T}^{S^*}\frac{K - S_T}{K^2}dK = \frac{S_T - S^*}{S^*} + ln\frac{S^*}{S_T}$$

Payoff of the portfolio is $\frac{S_T - S^*}{S^*} - ln\frac{S_T}{S^*}$ regardless of S_T. Choose $S^* = S_0$, then the payoff becomes $\frac{S_T - S_0}{S_0} - ln\frac{S_T}{S_0}$. The 2nd position is replicated by

- The portfolio $\int_0^{S_0}\frac{1}{K^2}P(K,T)dK + \int_{S_0}^{\infty}\frac{1}{K^2}C(K,T)dK$
- $\frac{1}{S_0}$ short position in the forward on the stock with delivery price S_0.

(d)

(i) Assess the effectiveness of the Vega hedge.

(ii) Suggest two potential ways to improve the hedge.

Commentary on Question:
Candidate did well on this question. Most candidates successfully identified the limitation of BSM pricing formula and recommended reasonable approaches to overcome the ineffectiveness in Vega hedge.

11. **Continued**

(i) The Vega hedge is not very effective and could produce results worse than without the hedge. The reason is that a change in the current volatility under the Heston model would mean a change in the short term volatility and a much smaller change in the long term volatility, due to the mean reversion property of the model. If the option's 1st order derivative with respect to variance (unmodified vega) under the BSM model is used to set up the vega hedge, since volatility under BSM is constant, the volatility risk would be largely overestimate, resulting in ineffective hedge results.

(ii) One way to improve the hedge effectiveness is to use the modified vega of the guarantees under BSM when setting up the hedge.

$$Modified\,Vega = \sum_{t=\tau}^{T} v_t \frac{1}{\sqrt{t-\tau}}$$

Where τ is valuation time point and v_t represents the BSM Vega of each guaranteed cash flow in the future.

Another way to improve the effectiveness would be to determine the hedge position under the Heston model, which improves the modeling of the true dynamics of variance.

Partial points are also awarded if risk reversal and butterfly method are mentioned.

12. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1d) Understand and apply Ito's Lemma.

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

Sources:

Fixed Income Securities, Neftci

Commentary on Question:

This question is to test candidates on how to apply Ito's Lemma and the concept of Martingale.

Solution:

(a) Show, using Ito's Lemma, that

$$\frac{d\tilde{V}}{\tilde{V}} = (\sigma_Z^2 - \sigma_V \sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

Commentary on Question:

Overall, candidates did well on this part. Alternative approaches such as using product rule, quotient rule, etc. were also awarded full marks provided derivation was done correctly.

From Ito's lemma it's straightfoward to have

$$d(\ln V) = \left(r - \frac{\sigma_V^2}{2}\right)dt + \sigma_V dX_t$$

$$d(\ln Z) = \left(r - \frac{\sigma_Z^2}{2}\right)dt + \sigma_Z dX_t$$

It follows that

$$d\left(\ln\tilde{V}\right) = d(\ln V) - d(\ln Z) = \left(\frac{\sigma_Z^2}{2} - \frac{\sigma_V^2}{2}\right)dt + (\sigma_V - \sigma_Z)dX_t$$

12. Continued

In turn, from Ito's lemma it's straightfoward to have

$$\frac{d\widetilde{V}}{\widetilde{V}} = \left(\left(\frac{\sigma_Z^2}{2} - \frac{\sigma_V^2}{2}\right) + \frac{(\sigma_V - \sigma_Z)^2}{2}\right)dt + (\sigma_V - \sigma_Z)dX_t$$
$$= (\sigma_Z^2 - \sigma_V\sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

(b) Show that $\widetilde{V}$ is a martingale under $\mathbb{Q}^Z$ using:

(i) the result in part (a);

(ii) the Feynman-Kac theorem.

Commentary on Question:
Candidates did well on part (b)(i). However, very few candidates were able to apply Feynman-Kac theorem correctly. Partial marks were awarded for stating the theorem and identifying R(r)=0.

(i) By differntiating $\tilde{X}_t = X_t - \int_0^t \sigma_Z(r,u)du$

$$\widetilde{dX}_t = dX_t - \sigma_Z(r,t)dt,$$
$$dX_t = \widetilde{dX}_t + \sigma_Z(r,t)dt$$

plugging it in the dX_t of result of (a) yields the driftless martingale.

$$\frac{d\tilde{V}}{\tilde{V}} = (\sigma_Z^2 - \sigma_V\sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$
$$= (\sigma_V - \sigma_Z)\widetilde{dX}_t$$

(ii) From the Feynman-Kac theorem, it implies $R = 0$ in the following equation

$$R(r)\tilde{V} = 0 = \frac{\partial\tilde{V}}{\partial t} + \frac{\partial\tilde{V}}{\partial r}\,(m^*(r,t) + \sigma_Z\,(r,t)\,s(r,t)) + \frac{1}{2}\frac{\partial^2\tilde{V}}{\partial r^2}\,s(r,t)^{\,2}$$

Hence

$$\tilde{V}(r,t;\,T) = E_f^*\left(e^{-\int_t^T R(u)du}\;\tilde{V}(r,T;T)|r_t\right) = E_f{}^*\big(\tilde{V}(r,T;T)|r_t\big),$$

which is a martingale.

(c) Derive expressions for σ_z and σ_v in terms of $s(r,t)$, V, and Z.

Commentary on Question:
Candidates did very poorly on this part.

12. Continued

By the Ito's lemma and the Fundamental Pricing equation,

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} m^*(r,t) \; + \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial r^2} \; s(r,t)^2\right) dt + \frac{\partial V}{\partial r} s(r,t) dX_t$$
$$= rV dt + \frac{\partial V}{\partial r} s(r,t) dX_t$$
$$= rV dt + \sigma_V V dX_t$$

with

$$\sigma_V = \frac{1}{V}\left(\frac{\partial V}{\partial r}\right) s(r,t).$$

For other security $Z(r,t)$,

$$dZ = \left(\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} m^*(r,t) \; + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2} \; s(r,t)^2\right) dt + \frac{\partial Z}{\partial r} s(r,t) dX_t$$
$$= rZ dt + \sigma_Z Z dX_t$$

with

$$\sigma_Z = \frac{1}{Z}\left(\frac{\partial Z}{\partial r}\right) s(r,t).$$

13. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Understand and apply various one-factor interest rate models.

(3c) Calibrate a model to observed prices of traded securities.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

Sources:

Pietro Veronesi – Fix Income Securities Ch 15

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Interpret the meanings of the three parameters σ, γ, and $\bar{r}$.

Commentary on Question:

Candidates did well in this question. Most candidates demonstrated a good understanding of the meaning of the three parameters.

The physical meaning of the 3 Vasicek parameters $(\sigma, \gamma, \bar{r})$:

- σ: the measure of randomness of the interest rate;
- $\bar{r}$: the ultimate mean value of the interest rate;
- γ: the measure of coverging speed of the interest rate to $\bar{r}$;

13. **Continued**

(b) Recommend a method to calibrate the five parameters σ, γ, γ^*, $\bar{r}$ and $\bar{r}^*$.

Commentary on Question:
This question was designed to test the understanding of the parameters in the model with the real-world measure and the model with the risk-neutral measure. Most candidates knew the real-world parameters can be calibrated using historical data. Many candidates lost partial mark due to lack of details or not be able to point out that the risk neutral parameters can be calibrated based on bond market prices.

The volatility σ can be estimted directly by taking the standard deviation of the first difference in the time series of interest rate r_t;
$\bar{r}$ can be calculated as the average of short term interest rate r_t;
γ can be estimated by regressing the changes in $r_{t+\delta} - r_t$, where δ is the time between observations;
Similaraly, $(\gamma^*, \bar{r}^*)$ can be estimated using a time series of over night government bond instead of a time series of interest rate r_t.

(c) Calculate:

(i) the spot rate duration of a zero-coupon bond $Z(r,0;1)$;

(ii) the market value of the interest rate risk;

(iii) the risk premium associated with $Z(r,0;1)$.

Commentary on Question:
The question was to test the application of Vasicek model. Generally speaking, candidates did well in this question. Candidates did not get full mark due to mistake in calculation or the uncomplete answer.

Spot rate duration: $D_z(\tau) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^* * \tau}\right) = 0.98$
MV (Int rate risk): $\lambda(r,t) = \frac{1}{\sigma}\left(\gamma(\bar{r} - r) - \gamma^*(\bar{r}^* - r)\right) = -0.024$
Risk Premium: $-B(0,T) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^* * \tau}\right) = 0.98$

$$-B(0,T)\left(\gamma(\bar{r} - r) - \gamma^*(\bar{r}^* - r)\right) = 0.001176$$

(d) Describe how the spot rate duration depends on $\bar{r}^*$ and time to maturity τ.

Commentary on Question:
Candidates did well on this question.

13. Continued

From the formula of the spot duration, the spot rate duration is positively correlated with the maturity τ.
It doesn't depend on $\bar{r}^*$.

(e) Explain the behavior of the spot rate duration if $\bar{r}^* > 0$.

Commentary on Question:
Candidates did well on this question.

The spot duration is independent of the long-term mean $\bar{r}^*$.

(f) Calculate the zero-coupon bond price $Z(r_0, 0;1)$, assuming r_0 =1%.

Commentary on Question:
This question was to test how to apply the zero-coupon bond price formula derived from Vasicek model to compute bond price. Many candidates provided the answered in part (c) and got full mark as well. A common error is the miscalculation of A(0,1).

$Z(0,1) = e^{A(0,1)-B(0,1)*r}$,
Where

$$B(0,1) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*}\right) = 0.9803$$

$$A(0,1) = (B(0,1) - 1)\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^{*2}}\right) - \frac{\sigma^2 B(0,1)^2}{4\gamma^*}$$

$$= -0.0006.$$

Therefore, $Z(0,1) = e^{-0.0006-0.9803*0.03} = 0.9896$

14. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3b) Understand and apply various one-factor interest rate models.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3e) Demonstrate understanding of option pricing theory and techniques for interest rate derivatives.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch 15

Commentary on Question:

Most candidates did not do well on this question. Some candidates received partial credits and very few candidates received full credits for some parts but no candidates received full credits for part (e).

Solution:

(a) Critique the suggestions and recommend alternative methods as needed.

Commentary on Question:

Some candidates received full credits by understanding the underlying assumption of Vasicek model by explaining the parameter estimation correctly. Some received partial credits but many did not correctly answer the questions.

Suggestion 1:
Vasicek is a short-term interest rate model and thus the σ cannot be estimated from bond yield.
Recommendation: It should be estimated from the market data of short-term interest rate.

14. Continued

Suggestion 2:
The average short-term rate gives an estimate of long term mean and the regression on interest rate changes gives an estimate of rate of mean reversion in the real world measure. The two parameters describe the interest rate process under Vasicek model in real world but not the parameters that satisfy the fundamental pricing equation.
Recommendation: The $\bar{r}^*$ and γ^* should be estimated from non-linear least square technique that minimizes the difference between bond price suggested by Vasicek model and the market price of bonds.

(b) Calculate *Call*(2)

Commentary on Question:
Some candidates received partial credits by writing the formula correctly. Very few candidates received full credits by calculating the number totally correct.

$$Call(2) = \mathrm{Z}(\mathrm{r}_0, 0;\ 2) * N\big(d_1(2)\big) - K_2 * \ \mathrm{Z}(\mathrm{r}_0, 0;\ 1) * N\big(d_2(2)\big)$$

So first need to calculate $\mathbf{Z(r_0, 0;\ 1)}$:

By Vasicek model:

$$\mathrm{Z}(\mathrm{r}_0, 0;\ 1) = e^{(A(0;1) - B(0;1) * r_0)}$$

Recall that
$A(t;T) = A(0;T-t)$ and $B(t;T) = B(0;T-t)$

The corresponding value can be derived from
$A(1.0;2.0) = -0.010$ and $\mathrm{B}(1.0;2.0) = 0.805$

$$\mathrm{Z}(\mathrm{r}_0, 0;\ 1) = e^{(-0.010 - 0.805 * 0.01)} = 0.982$$

$$Call(2) = \mathrm{Z}(\mathrm{r}_0, 0;\ 2) * N\big(d_1(2)\big) - K_2 * \ \mathrm{Z}(\mathrm{r}_0, 0;\ 1) * N\big(d_2(2)\big)$$
$$= 0.954 * 0.637 - 0.965 * 0.982 * 0.6296 = 0.0111$$

(c) Calculate $K_{3.5}$

Commentary on Question:
Very few candidates received partial credits by writing the formula. Only a handful of them received full credits by solving the number correctly.

14. Continued

Recall that by definition, $K_i = \mathrm{Z}(\mathrm{r}_K, \mathrm{T}_O;\ T_i)$ where r_K is the interest rate that satisfies equation $\mathrm{P}(\mathrm{r}_K, \mathrm{T}_O;\ T_B) = 1$ or $100 * \mathrm{P}(\mathrm{r}_K, \mathrm{T}_O;\ T_B) = 100$

$$\mathrm{P}(\mathrm{r}_K, \mathrm{T}_O;\ T_B) = \sum_{i=1.5}^{4} \frac{c}{2} \mathrm{Z}(\mathrm{r}_K, \mathrm{T}_O;\ T_i) * 100 + \mathrm{Z}(\mathrm{r}_K, \mathrm{T}_O;\ T_4) * 100$$
$$= \sum_{i=1.5}^{4} \frac{c}{2} K_i * 100 + K_4 * 100 = 100$$

Therefore to solve for $K_{3.5}$

$$(0.9831 + 0.965 + 0.9462 + 0.9269 + 0.8877) * 0.02 * 100 + 0.8877 * 100 + 0.02 * 100 * K_{3.5} = 100$$

$$K_{3.5} = \frac{100 - 88.77 - (0.9831 + 0.965 + 0.9462 + 0.9269 + 0.8877) * 2 * 100}{2}$$
$$= 0.906$$

(d) Calculate the price of the embedded call option.

Commentary on Question:
Very few candidates received partial credits by writing the formula. Only a handful of them received full credits by calculating the number correctly.

The price of the embedded call

$$\mathrm{C} = 100 * (\frac{c}{2} \sum_{i=1.5}^{4} Call(i) + Call(4))$$

where

$$Call(i) = \mathrm{Z}(\mathrm{r}_0, 0;\ T_i) * N\big(d_1(i)\big) - K_i * \mathrm{Z}(\mathrm{r}_0, 0;\ T_O) * N\big(d_2(i)\big)$$

With all the components calculated, we can calculate the call option price:

$$\mathrm{C} = 100 * \left(\frac{c}{2} \sum_{i=1.5}^{4} Call(i) + Call(4)\right)$$
$$= \frac{0.04}{2} * (0.622 + 1.11 + 1.475 + 1.754 + 2.07 + 2.108) + 2.108$$
$$= 2.29$$

14. Continued

(e) Calculate the current price of the callable bond.

Commentary on Question:
Very few candidates received partial credits by writing part of the formula. No candidates received full credits by calculating the number correctly.

The price of the non-callable coupon bond is

$$P(r_0, 0;\ 5) = \sum_{i=0.5}^{4} \frac{c}{2} Z(r_0, 0;\ T_i) * 100 + Z(r_0, 0;\ 4) * 100$$

Where Z(r_0,0; 0.5) is yet to be calculated:

$$Z(r_0, 0;\ 0.5) = Z(r_0, 1;\ 1.5) = 0.993$$

Now we have:
$Z(r_0, 0;\ 0.5) = 0.993$ Given
$Z(r_0, 0;\ 1) \quad = 0.982$ Calculated in part (b)
$Z(r_0, 0;\ 1.5) = 0.969$ Given
$Z(r_0, 0;\ 2.0) = 0.954$ Given
$Z(r_0, 0;\ 2.5) = 0.938$ Given
$Z(r_0, 0;\ 3.0) = 0.921$ Given
$Z(r_0, 0;\ 3.5) = 0.903$ Calculated in part (b)
$Z(r_0, 0;\ 4.0) = 0.884$ Given

$P(r_0, 0;\ 4) = 2 * (0.993 + 0.982 + 0.969 + 0.954 + 0.938 + 0.921 + 0.903 + 0.884) + 100 * 0.884 = \ 103.49$

The value of the callable bond is the straight bond – embedded call option, since the bond give the issuer right to call the bond. Therefore, the value of the callable bond is

$$103.49 - 2.29 = 101.2$$

15. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.

(2b) Bootstrap a yield curve.

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Show that the one-year, two-year, and three-year zero-coupon rates are 5.5%, 6%, and 7%, respectively.

Commentary on Question:
Many candidates got the right answer here, and this was the most likely part to be totally correct.

$$1{,}042.65 = \frac{1{,}100}{1+s_1} \Rightarrow s_1 = \boxed{0.055}$$

$$1{,}073.78 = \frac{100}{1.055} + \frac{1{,}100}{(1+s_2)^2} \Rightarrow s_2 = \boxed{0.06}$$

$$1{,}081.71 = \frac{100}{1.055} + \frac{100}{1.06^2} + \frac{1{,}100}{(1+s_3)^3} \Rightarrow s_3 = \boxed{0.07}$$

(b) Calculate the fixed swap rate.

Commentary on Question:
Many candidates skipped this question and only ~ 25% who attempted it got it all right

$$R = \frac{1-P_3}{P_1+P_2+P_3} = \frac{1-\frac{1}{1.07^3}}{\frac{1}{1.055}+\frac{1}{1.06^2}+\frac{1}{1.07^3}}$$

$$[1pt]\ \boxed{= 0.0692129}$$

15. Continued

(c) Calculate the value of the fund at the end of three years.

Commentary on Question:
Majority of the candidates skipped this question, and very few got it all right. Some candidates got the formula of the future value of the fund at the end of year 3 right, even though the result at the end of year 1-3 may be incorrect.

Time 1: $10{,}000{,}000(0.0692129 - 0.055) = 142{,}128.63$
Time 2: $10{,}000{,}000(0.0692129 - 0.06) = 92{,}128.63$
Time 3: $10{,}000{,}000(0.0692129 - 0.07) = -7{,}871.37$
The value of the fund at the end of three years
$= 142{,}128.63(1.03^2) + 92{,}128.63(1.03^1) - 7{,}871.37$
$= \boxed{237{,}805.38}$

(d) Determine if an arbitrage opportunity is available. If it is available, design a strategy to exploit the arbitrage opportunity.

Commentary on Question:
Many candidates skipped this question and nobody received full mark. Most candidates got partial marks on the concept of Put-call parity, and some got additional marks on correctly applying it. Some skipped this question.

$$\mathrm{Call}(0.90) - \mathrm{Put}(0.90) = Z(0{,}2) \times \left(P^{\mathrm{fwd}}(0{,}2{,}3) - 0.90\right)$$
$$0.1431 - \mathrm{Put}(0.90) = \frac{1}{1.06^2} \times \left(\frac{1/1.07^3}{1/1.06^2} - 0.90\right)$$
$\mathrm{Put}(0.90) = 0.1278\ (> 0.10)$
Since the market put is underpriced, arbitrage opportunity is available.
To exploit arbitrage:

- buy low by buying the actual put
- sell high by selling the synthetic put

A short synthetic put can be created with a short call, a long forward, and a short bond.

(e) Describe two advantages and two disadvantages of futures over forwards.

Commentary on Question:
Most candidates answered this question. Some candidates got full or close to full points. Candidates got, on average, partial credit for this one. Some overlooked a key word in the question of "describe" and some just listed a one-word advantages / disadvantages instead, which received partial marks.

15. Continued

Advantages of using futures over forward:

- **Liquidity**. Because of their standardization, futures are more liquid than forward contracts, meaning that it is easy to get in and out of the position. For the highly traded futures contract, going in and out of positions is relatively inexpensive. This is not true for forward contracts. Because they are traded only over-the-counter, closing a position may be expensive.
- **Credit Risk**. The existence of a clearinghouse guarantees performance on futures contracts, while the same may not be true for forward contracts.

Disadvantages of using futures over forward:

- **Basis risk.** The available maturity of the bond or the particular instrument may not be the exact instrument to hedge all of the risk. Using a forward rate agreement, a firm could perfectly hedge the risk. Using futures, the firm would retain some residual risk, as the available instruments are not perfectly correlated with the interest rate to hedge.
- **Tailing of the Hedge.** The cash flows arising from the futures position accrue over time, which implies the need of the firm to take into account the time value of money between the time at which the cash flow is realized and the maturity of the hedge position. This will call for a reduction in the position in futures.