ALTAM April 2023 Model Solutions

Question 1

(a)

$$ {}_2p_x^{01} = \begin{pmatrix} 0.80 & 0.10 & 0.05 & 0.05 \end{pmatrix} \begin{pmatrix} 0.10 \\ 0.50 \\ 0.00 \\ 0.00 \end{pmatrix} = 0.130 $$

$$ {}_2p_x^{03} = \begin{pmatrix} 0.80 & 0.10 & 0.05 & 0.05 \end{pmatrix} \begin{pmatrix} 0.05 \\ 0.20 \\ 0.30 \\ 1.00 \end{pmatrix} = 0.125 $$

$$ {}_2p_x^{13} = \begin{pmatrix} 0.20 & 0.50 & 0.10 & 0.20 \end{pmatrix} \begin{pmatrix} 0.05 \\ 0.20 \\ 0.30 \\ 1.00 \end{pmatrix} = 0.340 $$

$$ {}_2p_x^{22} = \begin{pmatrix} 0 & 0 & 0.7 & 0.3 \end{pmatrix} \begin{pmatrix} 0.05 \\ 0.10 \\ 0.70 \\ 0 \end{pmatrix} = 0.490 $$

$$ {}_2p_x^{23} = \begin{pmatrix} 0 & 0 & 0.7 & 0.3 \end{pmatrix} \begin{pmatrix} 0.05 \\ 0.20 \\ 0.30 \\ 1.00 \end{pmatrix} = 0.510 $$

(b) The probability is equal to

a. the probability that one healthy life dies, and the other healthy lives and the permanently disabled life all survive

b. <u>plus</u> the probability that all healthy lives survive but the permanently disabled life dies;

That is:

$$\binom{14}{1}\,{}_2p^{03}\left(1-{}_2p^{03}\right)^{13}\left(1-{}_2p^{23}\right)+\left(1-{}_2p^{03}\right)^{14}\ {}_2p^{23}$$

$$=0.15113+0.07865=0.22977$$

(c) As the lives are independent, the expected number of deaths is

$$14\ {}_2p^{03}+{}_2p^{23}=2.26$$

(d)

(i) For the PD life:

A Death Benefit will be paid at time 1 with probability ${}_1p^{23}=0.3$

A Death Benefit will be paid at time 2 with probability ${}_1p^{22}\times{}_1p^{23}=0.21$

There are no remaining Disability Benefits

So the expected present value is

$$1{,}000{,}000\left(\frac{0.3}{1.06}+\frac{0.21}{1.06^2}\right)=469{,}918$$

(ii) For each of the 14 healthy lives:

A Death Benefit will be paid at time 1 with probability ${}_1p^{03}=0.05$

A Death Benefit will be paid at time 2 with probability ${}_2p^{03}-{}_1p^{03}=0.075$

A Disability Benefit is paid at time 1 with probability ${}_1p^{02}=0.05$

A Disability benefit is paid at time 2 with probability

${}_1p^{00}\ {}_1p^{02}+{}_1p^{01}\ {}_1p^{12}=0.05$

So the EPV of all benefits to all 14 healthy lives is

$$14\left(1{,}000{,}000\left(0.05v_{6\%}+0.075v_{6\%}^2\right)+500{,}000\left(0.05v_{6\%}+0.05v_{6\%}^2\right)\right)$$

$$=1{,}594{,}874+641{,}687=2{,}236{,}561$$

(e) Possible answers

a. We do not expect these values to be homogeneous – that is, we expect the transition probabilities to change over time.
b. We do not expect the probabilities to be Markov – for example, the probability of becoming permanently disabled for a life that is temporarily disabled might depend on the length of time that the life has been in the temporarily disabled state.

Examiners' Comments:

(a) *Candidates did well on this part. The most common mistake was calculation error.*

(b) *Although many candidates earned maximum credit, many did not. The most common errors included not allowing for the 14 healthy lives, or ignoring the permanently disabled life.*

(c) *Candidates did well on this part.*

(d) *Candidates had the most difficulty with the time 2 benefits. Some tried to use shortcuts, which missed states that a life could move into. A few candidates tried to use the part (i) answer in part (ii), not realizing they were unrelated benefits.*

(e) *Several candidates stated advantages instead of disadvantages. Some stated disadvantages of using an average instead of relating it to the homogeneity of the Markov model, as required in the question.*

Question 2

(a)

- A couple may have common lifestyles leading to similar mortality (eg both smokers)
- The broken heart syndrome increases mortality of the surviving partner in the period following their spouse's death
- The couple may be exposed to common shock risk, i.e., the risk that they die at the same time, e.g., in an accident.

(b) We require:

$\mu^{03}_{x+t:y+t} = 0$ (no common shock)

$\mu^{02}_{x+t:y+t} = \mu^{13}_{y+t}$ and $\mu^{01}_{x+t:y+t} = \mu^{23}_{x+t}$

(no common lifestyle or broken heart syndrome)

(c)

(i) $$\frac{d}{dt}\,{}_t p^{00}_{x:y} = -\,{}_t p^{00}_{x:y}\left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)$$

BC: ${}_0 p^{00}_{x:y} = 1.0$

(ii) $$\frac{d}{dt}\,{}_t p^{00}_{x:y} = -\,{}_t p^{00}_{x:y}\left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)$$

$$\Rightarrow \frac{1}{{}_t p^{00}_{x:y}}\;\frac{d}{dt}\,{}_t p^{00}_{x:y} = -\left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)$$

$$\Rightarrow \frac{d}{dt}\log\,{}_t p^{00}_{x:y} = -\left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)$$

$$\Rightarrow \int_0^n \frac{d}{dt}\log\,{}_t p^{00}_{x:y}\,dt = -\int_0^n \left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)dt$$

$$\Rightarrow \log\,{}_n p^{00}_{x:y} - \log\,{}_0 p^{00}_{x:y} = -\int_0^n \left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)dt$$

$$\log\,{}_0 p^{00}_{x:y} = \log 1.0 = 0$$

$$\Rightarrow {}_n p^{00}_{x:y} = \exp\left(-\int_0^n \left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t}\right)dt\right)$$

(d) $P\ddot{a}_{\overline{40:50}:\overline{10|}} = 100,000\left(A_{\overline{40:50}}\right)$

$\ddot{a}_{\overline{40:50}:\overline{10|}} = \ddot{a}_{40:\overline{10|}} + \ddot{a}_{50:\overline{10|}} - \ddot{a}_{40:50:\overline{10|}} = 8.1076$

$A_{\overline{40:50}} = A_{40} + A_{50} - A_{40:50} = 0.098737$

$\Rightarrow P = 1217.83$

(e) (i) ${}_{10}V^{(1)} = 100,000 A_{60} = 29,028$

(ii) ${}_{10}V^{(0)} = 100,000\left(A_{\overline{50:60}}\right) = 15,911.1$

(f)

(i) The policy value would stay the same.

Once (x) has died, the policy value only depends on the marginal mortality of (y), which is unchanged.

(ii) The policy value would increase.
The time to the first death is the same as under the independent model, but the time to the second death is less than or equal to the independent model, because of the possibility of simultaneous deaths.

Examiners' Comments:

(a) Candidates did well on this part. Note that full credit requires more than writing down key words.

(b) Candidates did well on this part, apart from some carelessness with subscripts.

(c) This part proved more challenging. Candidates were required to show a full coherent proof for full credit. Candidates lost points by skipping key steps (eg writing down the first and last line of the derivation, but failing to connect them), by failing to check the boundary condition, and by using confusing or missing subscripts and/or variables of integration.

(d) Generally well done. A few candidates used a joint life annuity instead of last survivor.

(e) This part was very well done.

(f) This part was generally well done. For full credit candidates were required to provide coherent explanations of how the policy values change based on the timing of the second death.

Question 3

(a)

(i) $AV_9 = \left(AV_8 + P(0.94) - 55 - (COI)_9\right)(1.06)$

where $AV_8 = 62,000; \quad P = 7000 \quad F = 150000$

$(COI)_9 = \max\left(0.002\left(F - AV_9\right)v_{5\%},\, 0.002(0.97)AV_9\ v_{5\%}\right)$

Assume (first) that $F - AV_9 > 0.97AV_9$ (i.e. CF does not apply)

$\Rightarrow (COI)_9 = 0.002\left(F - AV_9\right)v_{5\%}$

$\Rightarrow AV_9 = \left(62000 + 7000(0.94) - 55 - \left(0.002\left(F - AV_9\right)v_{5\%}\right)\right)(1.06)$

$$= \frac{72636 - 0.002v_{5\%}(150,000)}{1 - 0.002v_{5\%}(1.06)} = \frac{72333}{0.99798}$$

$= 72480$

Check CF condition: $F - AV_9 = 77521; \qquad 0.97AV_9 = 70305$

$\Rightarrow$ CF does not apply, as assumed.

(ii) $AV_{10} = \left(AV_9 + P(0.94) - 55 - (COI)_{10}\right)(1.06)$

where $AV_9 = 72480;$

$(COI)_{10} = \max\left(0.0022\left(F - AV_{10}\right)v_{5\%},\, 0.0022(0.97)AV_{10}\ v_{5\%}\right)$

Assume (first) that $F - AV_{10} > 0.91AV_{10}$,

$\Rightarrow (COI)_{10} = 0.0022\left(F - AV_{10}\right)v_{5\%}$

$\Rightarrow AV_{10} = \left(72480 + 7000(0.94) - 55 - \left(0.0022\left(F - AV_{10}\right)v_{5\%}\right)\right)(1.06)$

$$= \frac{83745 - 0.0022v_{5\%}(150,000)(1.06)}{1 - 0.0022v_{5\%}(1.06)} = \frac{83412}{0.99778}$$

$= 83598$

Check CF condition: $F - AV_{10} = 66402 < 0.91AV_{10} = 76074 \Rightarrow$ CF does apply.

Redo the calculation with

$$\Rightarrow (COI)_{10} = 0.0022(0.91)AV_{10}v_{5\%}$$

$$\Rightarrow AV_{10} = \left(72480 + 7000(0.94) - 55 - \left(0.0022(0.91)AV_{10}v_{5\%}\right)\right)(1.06)$$

$$= \frac{83745}{1 + (0.91)0.0022v_{5\%}(1.06)}$$

$$= 83576$$

(b) The death benefit at the end of year 10 is $1.91AV_{10}$

The associated death expense is 250.

The probability of death for year 10 is $q_{49} = 0.0011$

$$\Rightarrow \text{EDB}_{10} = \left(1.91AV_{10} + 250\right)q_{49} = 175.9$$

(c)

$$\text{Pr}_{10} = AV_9 + P - E_{10} + I_{10} - EDB_{10} - ESV_{10} - EAV_{10}$$

$$ESV_{10} = (1 - q_{49})(0.15)(AV_{10} - 1500 + 100) = 12,313$$

$$EAV_{10} = (1 - q_{49})(0.85)AV_{10} = 70,961$$

$$\text{Pr}_{10} = \left(72,480 + 7000(0.95) - 45\right)(1.075) - 176 - 12313 - 70961$$

$$= 1566$$

(d)

$$\Pi_{10} = {}_9p_{40}^{(\tau)}\,\text{Pr}_{10} = {}_9p_{40}\,(1 - 0.02)^9(1566) = 1297$$

(e)

1. The insurer may be required by regulations to hold more reserves.
2. The insurer may hold more reserves for no lapse guarantees.
3. The insurer may need to reserve for the potential additional costs arising from guaranteed maximum mortality or expense rates or guaranteed minimum credited interest rate.
4. The insurer may choose to hold additional reserves to allow for adverse experience, even if mortality, interest, or expense rates are changeable, to avoid having to change rates which could adversely impact policyholder behavior.

Examiners' Comments:

(a) Many candidates earned full credit. For those who did not, the most common error was failing to verify whether the corridor factor applied in each case.

(b) (c) Generally done well. The most common error involved using q_{50} instead of q_{49}.

(d) Generally done well.

(e) Only two reasons were required, but the answers needed to be relevant, reasonable and coherent for full credit.

Only the strongest candidates achieved full credit here, with most earning half credit for giving one correct reason.

Question 4

(a) The expected present value of future costs for (65) is

$$EPV = 12\left(3000\bar{a}_{65}^{00} + 7000\bar{a}_{65}^{01} + 15000\bar{a}_{65}^{02}\right)$$
$$= 592,180$$

(b) The parameter γ satisfies

$$EPV = 12\gamma\left(3,000\bar{a}_{65}^{00} + 7,000\bar{a}_{65}^{01} + 15,000\bar{a}_{65}^{02}\right) + 150,000$$
$$\Rightarrow \gamma = \frac{592,179.6 - 150,000}{592,179.6} = 0.7467$$

(c) $$ {}_5V_{65}^{(1)} = 12\left(7,000(1-\gamma)\bar{a}_{70}^{11} + 15,000(1-\gamma)\bar{a}_{70}^{12}\right)$$
$$= 261,917$$

(d)

(i) Thiele's equation for ${}_tV_{65}^{(1)}$ is

$$\frac{d}{dt}\,{}_tV_{65}^{(1)} = \delta \times {}_tV_{65}^{(1)} - 12\left(1-\gamma\right)\times 7000 - \mu_{65+t}^{12}\left({}_tV_{65}^{(2)} - {}_tV_{65}^{(1)}\right) - \mu_{65+t}^{13}\left(0 - {}_tV_{65}^{(1)}\right)$$

(ii)

$$\left.\frac{d}{dt}\,{}_tV_{65}^{(1)}\right|_{t=5} = \delta \times 261,917 - 12\left(1-0.7467\right)\times 7000$$
$$-0.00156\left(419,280 - 261,917\right) \;\; -0.011940\left(0 - 261,917.2\right)$$
$$= -5,616.5$$

(iii) Using Euler's forward method, we have

$${}_{5+h}V_{65}^{(1)} \approx {}_5V_{65}^{(1)} + h\left(\left.\frac{d}{dt}\,{}_tV_{65}^{(1)}\right|_{t=5}\right)$$
$$\Rightarrow {}_{5+h}V_{65}^{(1)} \approx 261,917.2 + 0.25\times\left(-5,616.54\right) = 260,513.1$$

(e)

The ILU residence fee is higher under the Full Life Care contract.

Both the EPV of costs and the entry fee are the same for an individual, so the EPV of total residence fees under the Full Life Care and modified life care are the same, but the modified life care fees step up on transition out of the ILU. The FLC effectively spreads that increase over the whole residency period, resulting in higher fees in the ILU.

Examiners' Comments:

Candidates generally did well on this question, with most candidates earning full credit for (a), (b) and (c). Part (d) was the most challenging section of the question, though many students received partial credit for (iii) even if the input values from (ii) were not correct. For (e), candidates were required to state coherently why ILU fees are higher under full life care contract to earn any credit. Most candidates handled this well, but a few provided long but irrelevant answers, earning little or no credit.

Question 5

(a) The accumulated amount is as follows, where $i=(1+0.06/12)^{12}-1=6.1678\%$

$$
\begin{aligned}
V = 60,000(0.11)\frac{1}{12}\Big\{&\Big((1+i)^{11/12}+(1+i)^{10/12}+\ldots+1\Big)(1+i)^{19} \\
&+(1.03)\Big((1+i)^{11/12}+(1+i)^{10/12}+\ldots+1\Big)(1+i)^{18} \\
&+(1.03)^2\Big((1+i)^{11/12}+(1+i)^{10/12}+\ldots+1\Big)(1+i)^{17} \\
&+\cdots\ +(1.03)^{19}\Big((1+i)^{11/12}+(1+i)^{10/12}+\ldots+1\Big)\Big\}
\end{aligned}
$$

$$=6,600s_{\overline{1}|}^{(12)}\left\{(1+i)^{19}+(1.03)(1+i)^{18}+\cdots+(1.03)^{19}\right\}$$

$$V=6,600s_{\overline{1}|}^{(12)}(1+i)^{19}\left(\frac{1-\left(1.03v\right)^{20}}{1-\left(1.03v\right)}\right)=322,128$$

where $s_{\overline{1}|}^{(12)}=\dfrac{i}{i^{(12)}}=1.02796$

(b) The projected final year salary is $S_{64}=60,000(1.03)^{39}=190,022$

The target income is therefore $0.65\times 190,022=123,514$

The net annuity after government benefits is $123,514-18,000=105,514$

(c) The funds needed at retirement are $105,514\times 13.60=1,434,990$.

The salary at age 45 is $S_{45}=60,000(1.03)^{20}=108,367$

Let C denote the total (employer and employee) contribution rate. Then the equation of value is

$$1,434,990=322,128(1+i)^{20}+S_{45}\ C\ s_{\overline{1}|}^{(12)}(1+i)^{19}\left(\frac{1-(1.03v)^{20}}{1-(1.03v)}\right)$$

$$=1,066,341+108,367(1.02796)(3.11790)(15.2285)C$$

$$=1,066,341+5,289,237C$$

$$\Rightarrow C=\frac{1,434,990-1,066,341}{5,289,237}=0.069698$$

As this is less than 8%, the contribution is split evenly between the employee and employer, each paying 3.485%.

(d) The 10-year guarantee increases the cost of the annuity, which will decrease the amount of the annuity for a fixed purchase price. Thus, the replacement ratio will decrease.

Examiners' Comments

(a) This question required candidates to recall the material from earlier exams on annuities-certain, or to work the problem out in Excel from first principles. It proved very challenging to many candidates.
Candidates who used Excel and recorded the correct answer, earned full credit (they were expected to note that they had used Excel in their answer sheets). If the value transcribed from Excel was incorrect, partial credit was only awarded if the candidate wrote down a correct formula on their answer sheet. Note that the graders do not have access to the candidate's excel workbook.
The most common error (other than Excel calculations) was to ignore the fact that payments are monthly and/or that increases are annual.

(b) This part was well done with most candidates earning full credit.

(c) This part was not well done, with many candidates skipping it, or offering only a minimal attempt. Candidates were expected to recognize that they should use the calculation (or given value) from (a), but only a minority did so.

(d) This part was generally well done.

Question 6

(a) A Guaranteed Minimum Maturity Benefit offers a guaranteed lump sum payment if the policyholder survives in force to the contract maturity date.
At maturity, if the policyholder's account value is less than the GMMB, the insurer must fund the difference.

A GMIB offers a guaranteed lifetime income at maturity. The amount of income is usually specified in terms of a Benefit Base (BB) and a guaranteed annuitization rate.
The guaranteed annuity rate multiplied by the BB gives the annual income under the guarantee. This is typically paid as a life annuity, often with a guaranteed minimum payment period. At maturity, the insurer must fund the difference, if greater than 0, between the value of the annuity benefit and the policyholder's account value.

The GMIB is typically added to a VA contract as a rider, additional to the GMMB. The policyholder can select whether to take the GMMB lump sum or the GMIB income at maturity.

(b)

(i) $\pi(0) = {}_{10}p_{50}\, p(0)$ where $p(0)$ is the put option price.

$${}_{10}p_{50} = l_{60} / l_{50} = 0.980297$$

$$p(0) = Ke^{-10r}\Phi(-d_2(0)) - S_0(0.94)(0.998)^{120}\Phi(-d_1(0))$$

$$K = 9000; \quad S_0(0.94)(0.998)^{120} = 7392.5; \quad \sigma = 0.25; \quad r = 0.04$$

$$d_1(0) = \frac{\log\left(7392.5 / K\right) + \left(r + \sigma^2 / 2\right)(10)}{\sigma\sqrt{10}} = 0.6524 \qquad d_2(0) = d_1(0) - \sigma\sqrt{10} = -0.1382$$

$$\Rightarrow p(0) = 9000e^{-0.4}\Phi(0.1382) - 7392.5\,\Phi(-0.6524)$$

$$= 9000e^{-0.4}(0.55496) - 7392.5(0.25708)$$

$$= 3348.0 - 1900.5 = 1447.5$$

$$\Rightarrow \pi(0) = 0.980297\left(3348.0 - 1900.5\right) = 3282.0 - 1863.0 = 1419.0$$

(ii) The bond portion is 3282.0 (i.e. the first term in $\pi(0)$)

(c) The cost of the hedge required at time 1 is

$$\pi(1) = {}_9p_{51}\, p(1) \text{ where } p(1) \text{ is the put option price.}$$

$${}_9p_{51} = l_{60} / l_{51} = 0.981483$$

$$p(1) = Ke^{-9r}\Phi(-d_2(1)) - S_1(0.94)(0.998)^{120}\,\Phi(-d_1(1))$$

$$S_1(0.94)(0.998)^{120} = 1.05S_0(0.94)(0.998)^{120} = 7762.2$$

$$d_1(1) = \frac{\log\left(7762.2 / K\right) + \left(r + \sigma^2 / 2\right)(9)}{\sigma\sqrt{9}} = 0.6577$$

$$d_2(1) = d_1(1) - \sigma\sqrt{9} = -0.09229$$

$$\Rightarrow p(1) = 9000e^{-0.36}\Phi(0.09229) - 7762.2\,\Phi(-0.6577)$$

$$= 9000e^{-0.36}(0.53677) - 7762.2(0.25536) = 1388.2$$

$$\Rightarrow \pi(1) = 0.98148\left(3370.4 - 1982.2\right) = 1362.5$$

The hedge brought forward (allowing for survival bonus) is

$$\left(3282.0e^{0.04} - 1863.0(1.05)\right)\frac{l_{50}}{l_{51}} = 1461.5$$

The rebalancing cost is therefore $1362.5 - 1461.5 = -99.0$ (gain)

Examiners' Comments:

This proved to be a challenging question for many candidates.

In (a), most candidates gave answers that were incorrect, or that were too brief for substantial credit. Many skipped this part.

In (b) most candidates earned substantial partial credit, but very few earned the maximum points. The 90% guarantee was omitted by many candidates, or was applied incorrectly. A few candidates did not realize that they needed to use Excel to determine the required Normal Distribution probabilities.

In (c), some candidates did not calculate the rolled forward value of the time 0 hedge portfolio, and few candidates recognized the need to adjust for the survival bonus. Many candidates skipped this part altogether.